An Efficient Dictionary for Reconstruction of Sampled Multiband Signals

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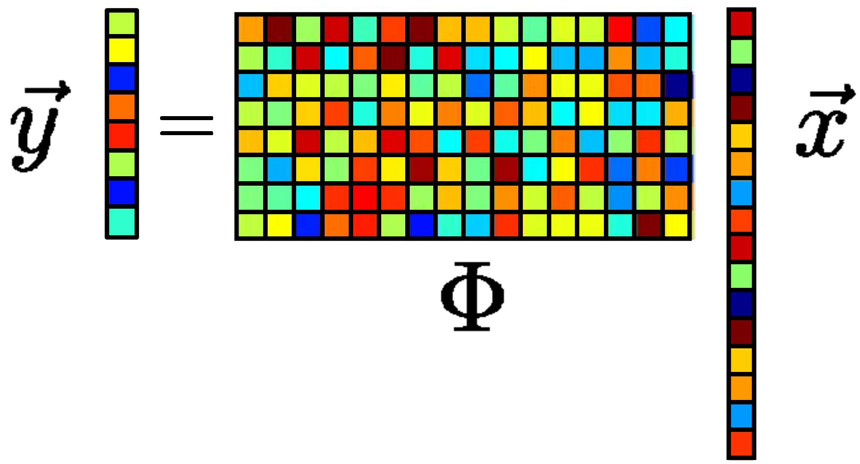
Two Regimes

Compressive Sensing (CS) is discrete-time, finite



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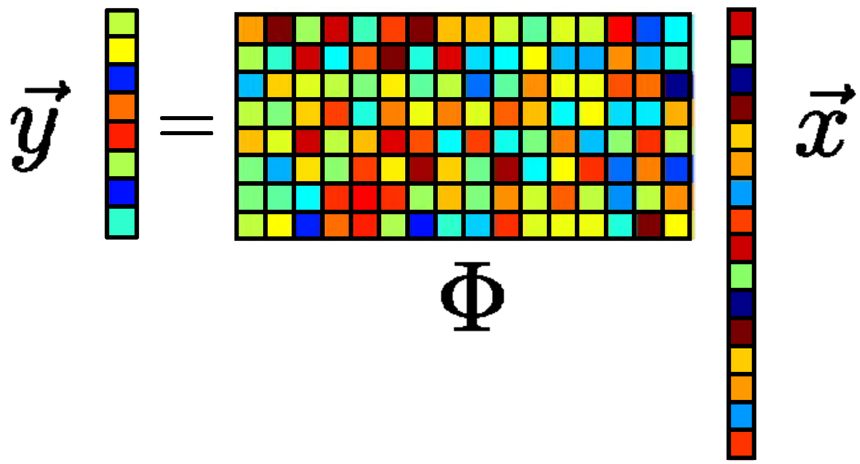
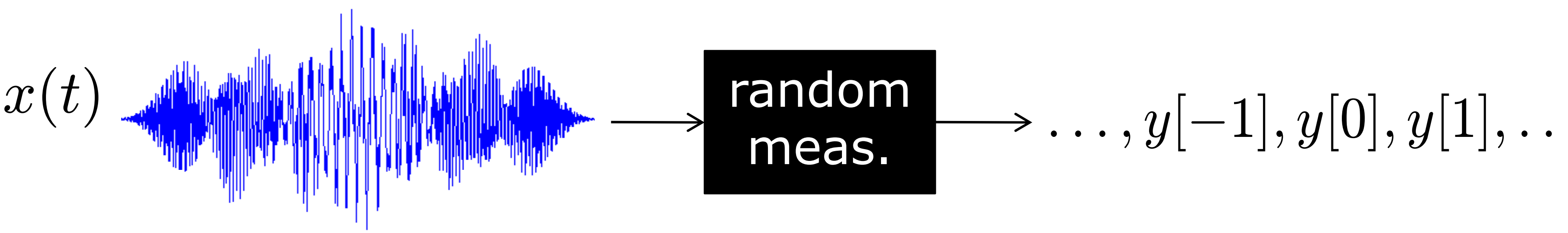


Analog signals are continuous-time, infinite



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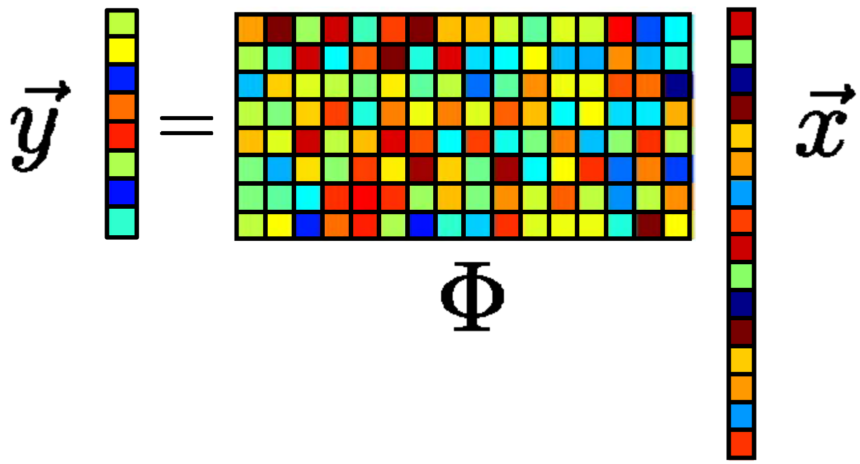

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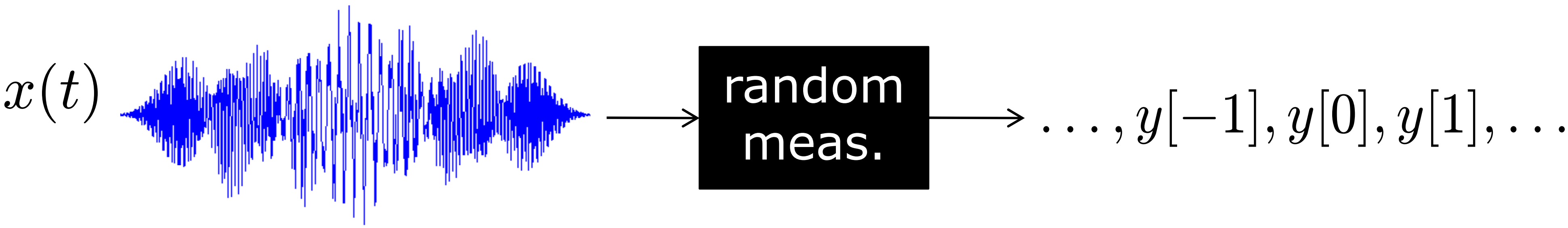

How compatible are these regimes?

Potential Challenges

Challenge 1:

Map analog sensing into matrix multiplication

Challenge 2:

Map analog sparsity into digital sparsity

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If $x(t)$ is bandlimited,

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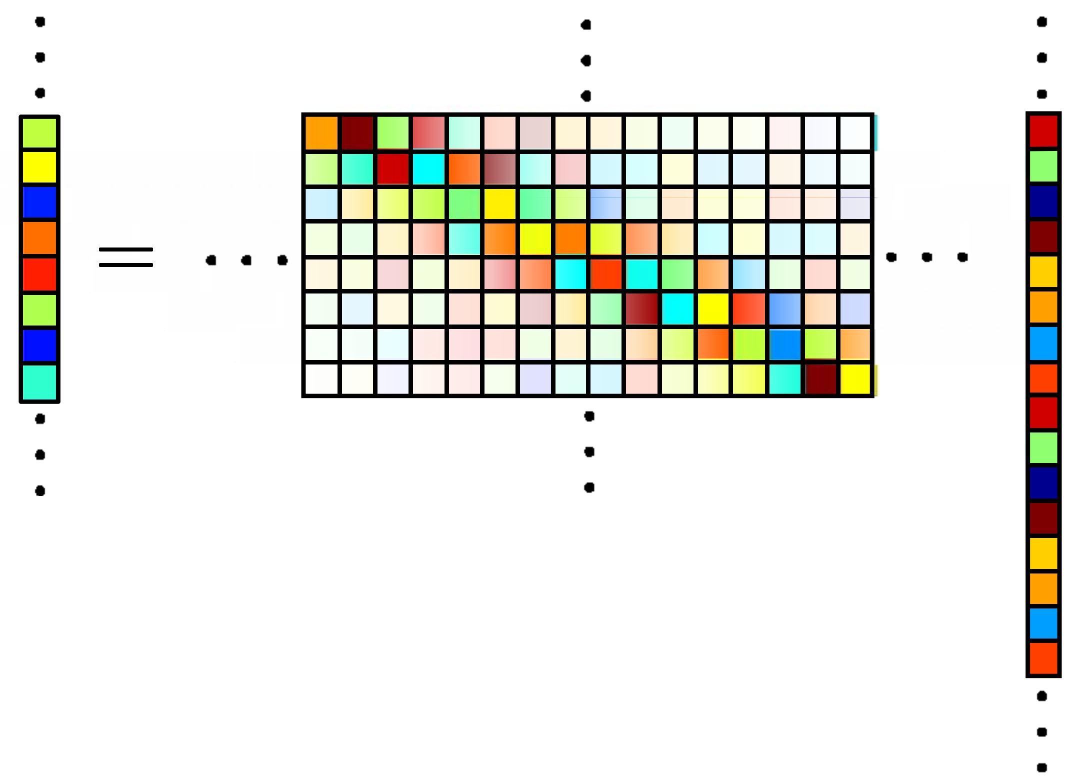

measurements

Nyquist-rate samples of $x(t)$

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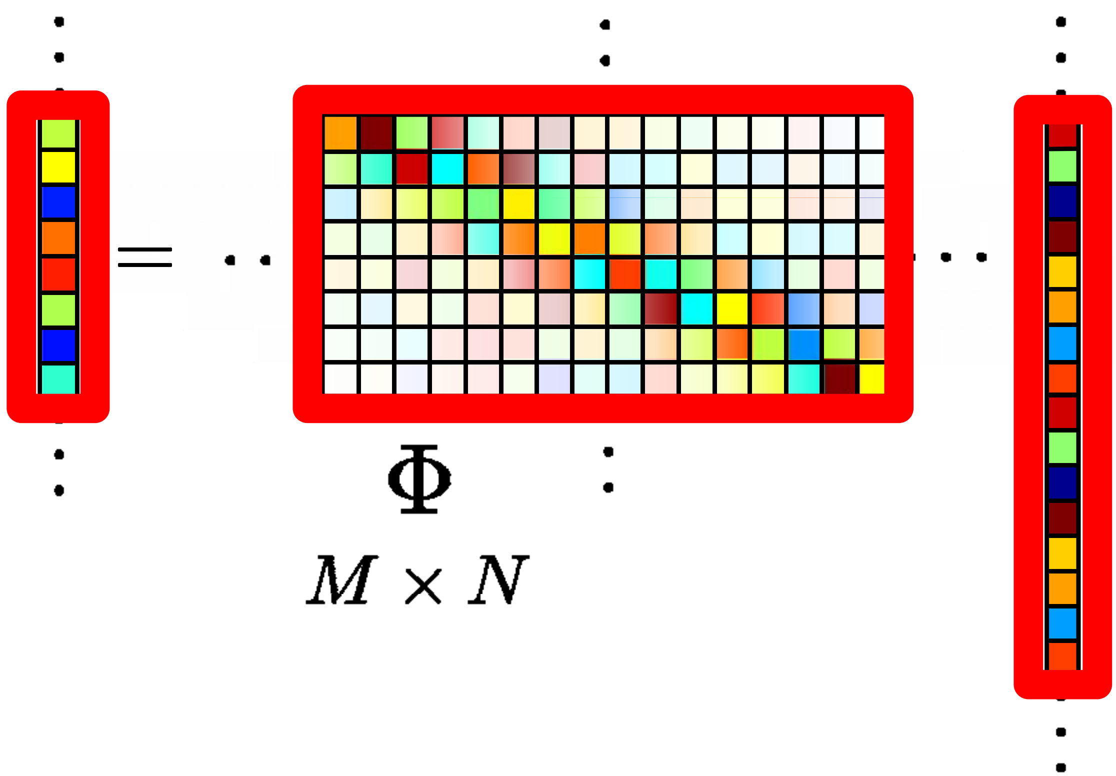

$\vec{y}$

$M \times 1$
measurements

Φ

$M \times N$

$\vec{x}$

$N \times 1$
vector

Nyquist-rate
samples
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Challenge 2:

Map analog sparsity into digital sparsity

$N \times 1$
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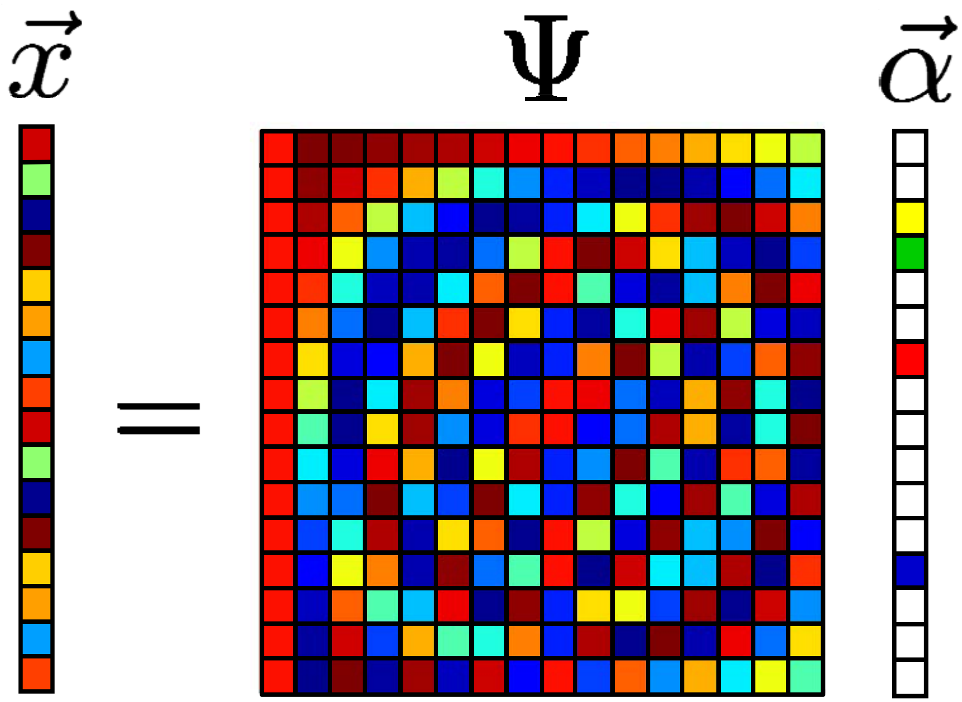
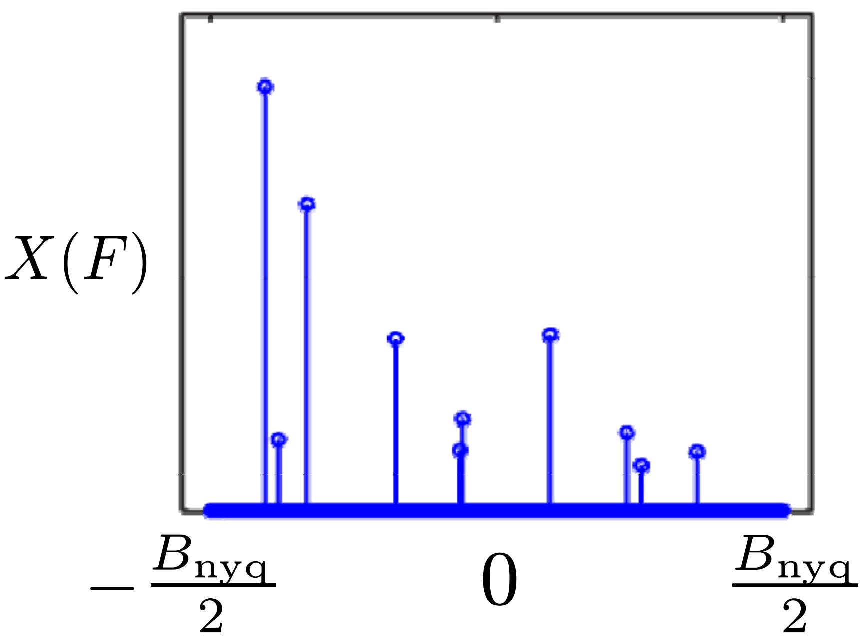


Candidate Models

	Model for $x(t)$	Basis for $\overrightarrow{x}$	Sparsity level for $\overrightarrow{x}$
multitone	sum of S "on-grid" tones	Ψ = DFT	S-sparse



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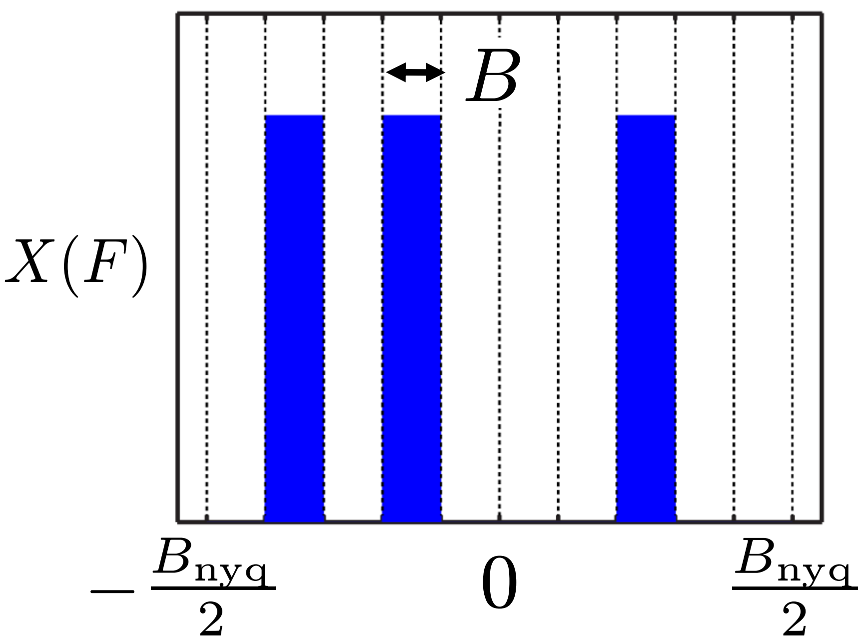
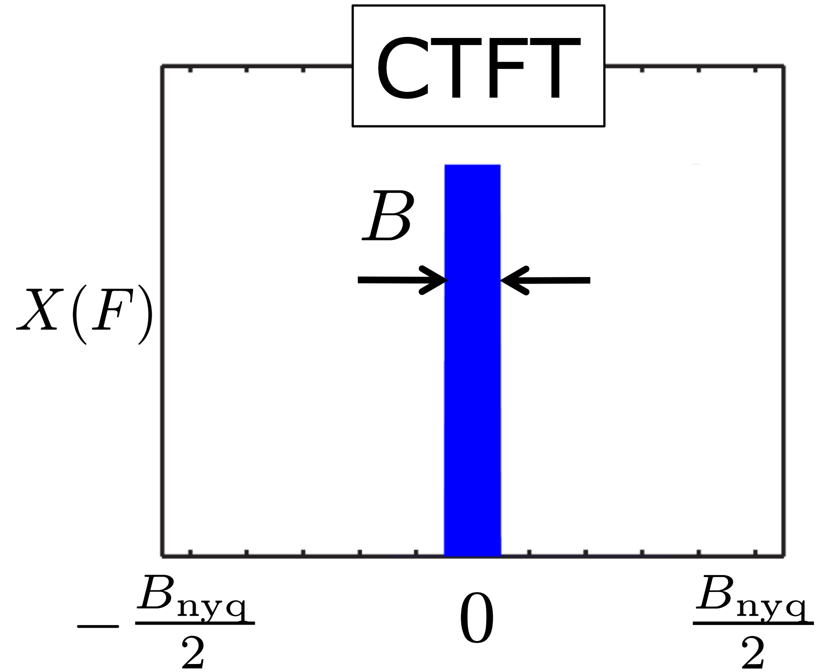
multiband	K occupied bands of bandwidth B	Ψ = ?	?



- Landau
- Bresler, Feng, Venkataramani
- Eldar, Mishali

The Problem with the DFT

$$x(t) = \int_{-\frac{B}{2}}^{\frac{B}{2}} X(F) e^{j2\pi Ft} \, dF$$



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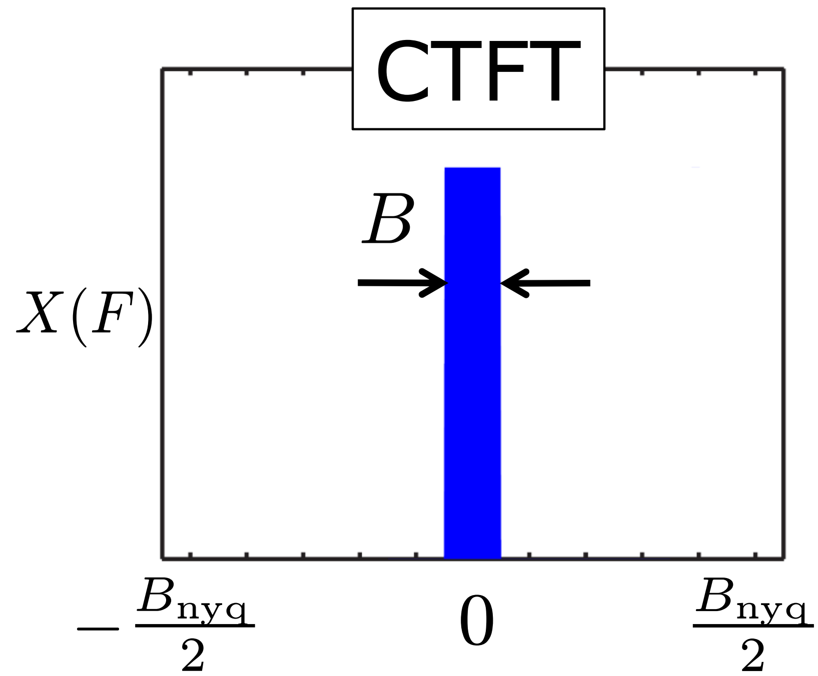
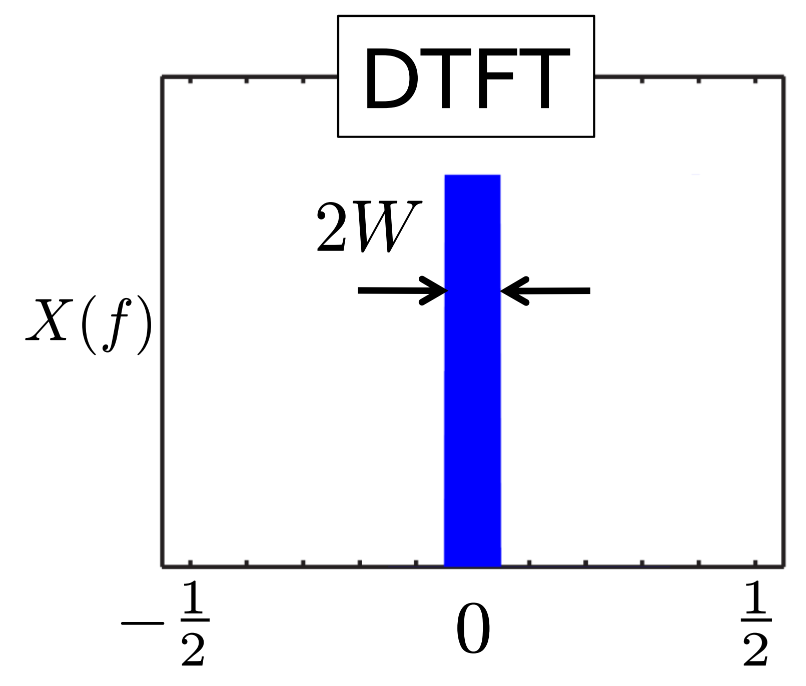
$$W = \frac{B}{2B_{\text{nyq}}}$$





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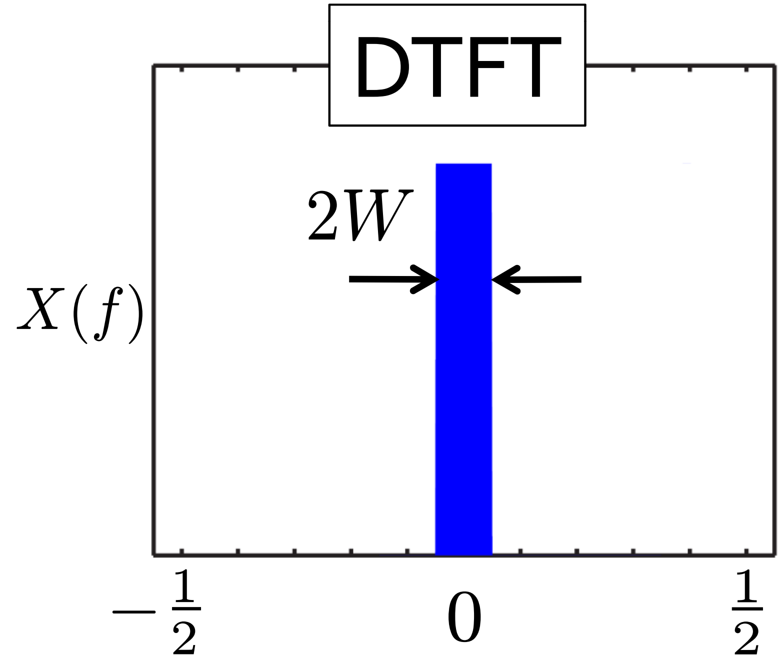

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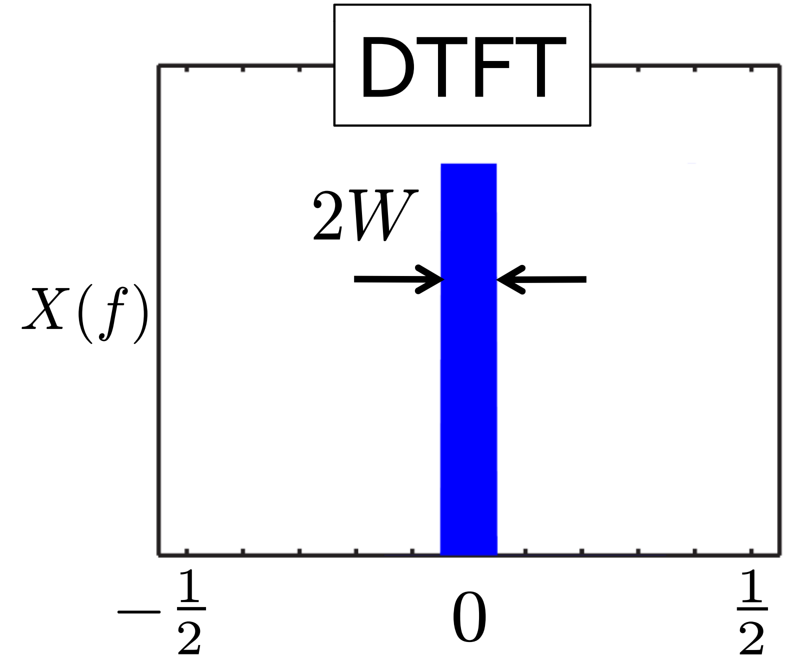
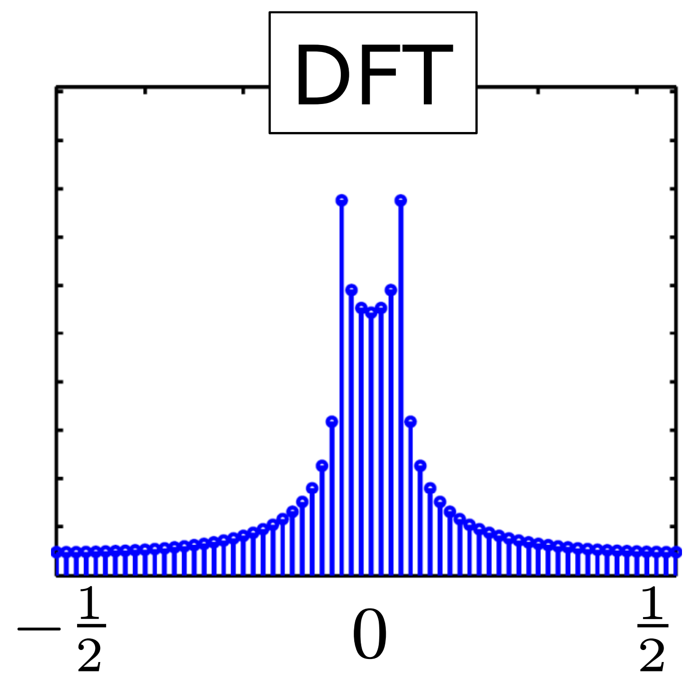


time-limiting

$$\vec{x} = \sum_{k=0}^{N-1} X_k \vec{e}_{\frac{k}{N}}, \quad \vec{e}_f := \begin{bmatrix} e^{j2\pi f 0} \\ e^{j2\pi f} \\ \vdots \\ e^{j2\pi f (N-1)} \end{bmatrix}$$



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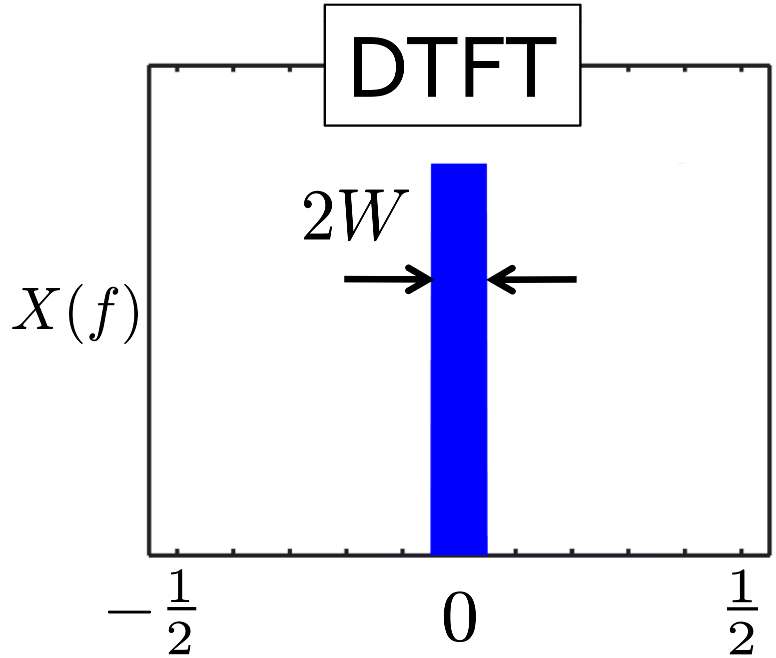

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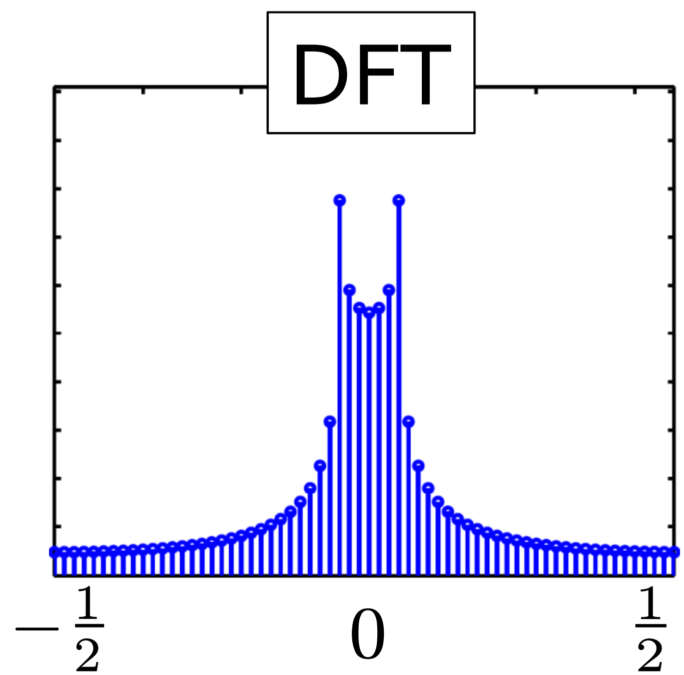

NOT SPARSE

Alternative Perspective

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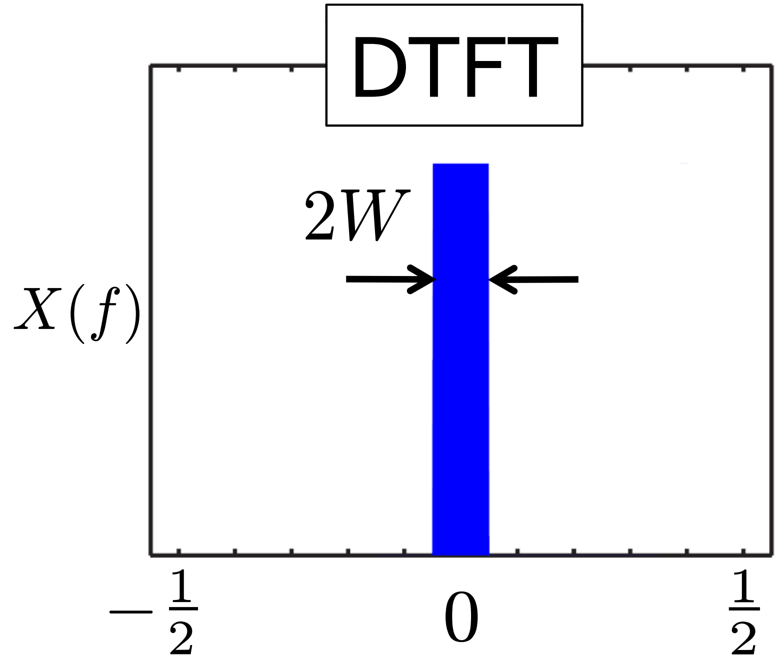

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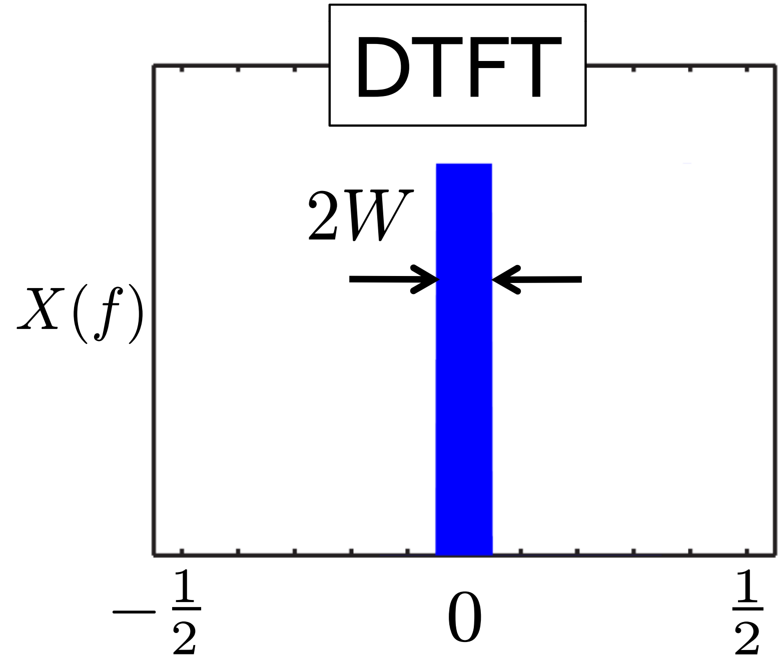

time-limiting

$$\mathcal{T}_N(x[n]) = \int_{-W}^{W} X(f) \mathcal{T}_N(e^{j2\pi f n}) \, df, \ \forall n$$

Building Blocks for Lowpass Signals

Time-limited complex exponentials form a "basis" for $\vec{x}$:

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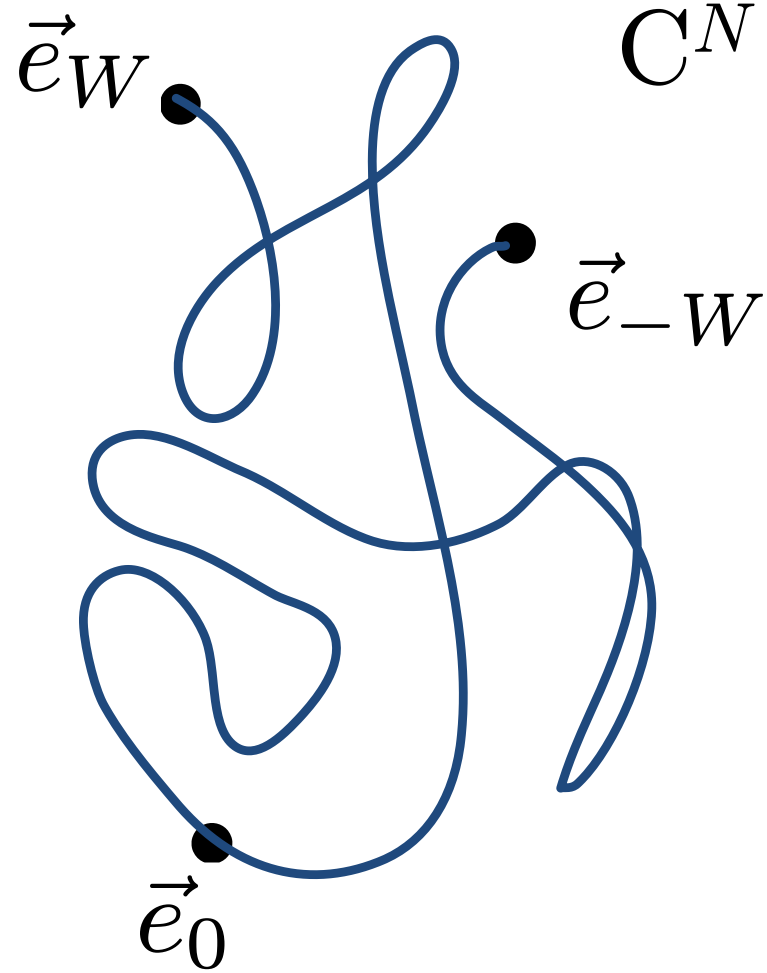

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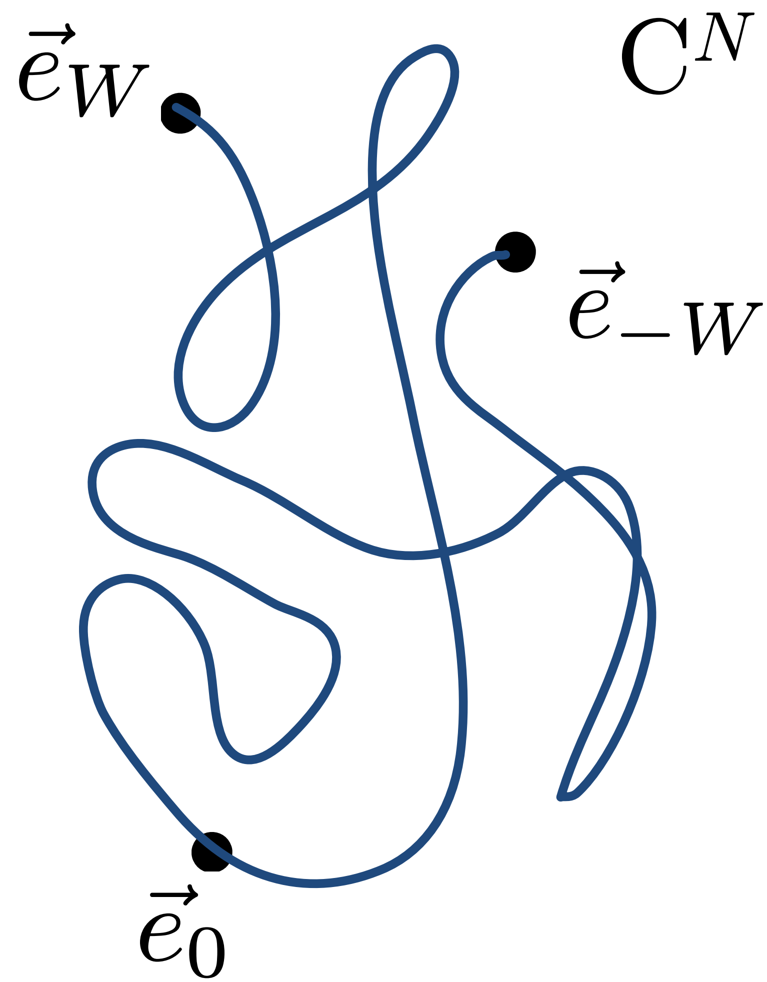

The problem: we need infinitely many of them.

Best Subspace Fit

Suppose that we wish to minimize

$$\int_{-W}^{W} \|\vec{e}_f - P_Q \vec{e}_f\|_2^2 \, df$$

over all subspaces Q of dimension k.

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Optimal subspace is spanned
by the first k “DPSS vectors”.

Discrete Prolate Spheroidal Sequences (DPSS's)

Slepian [1978]: Given an integer N and $W \leq 0.5$, the *DPSS's* are a collection of N vectors

$$\vec{s}_0, \vec{s}_1, \ldots, \vec{s}_{N-1} \in \mathbf{R}^N$$

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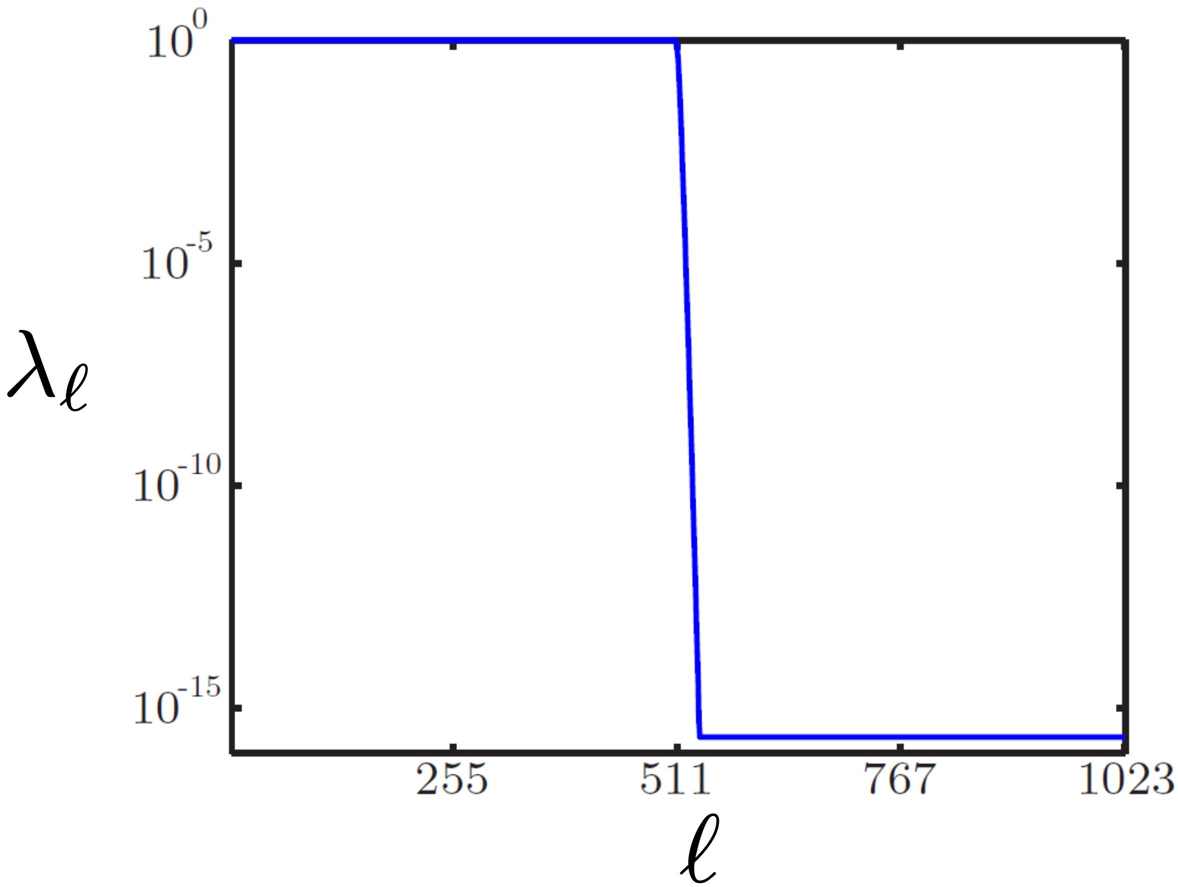
The DPSS's are perfectly time-limited, but when

$$\lambda_\ell \approx 1$$

they are highly concentrated in frequency.

DPSS Eigenvalue Concentration



$$N = 1024$$
$$W = \frac{1}{4}$$

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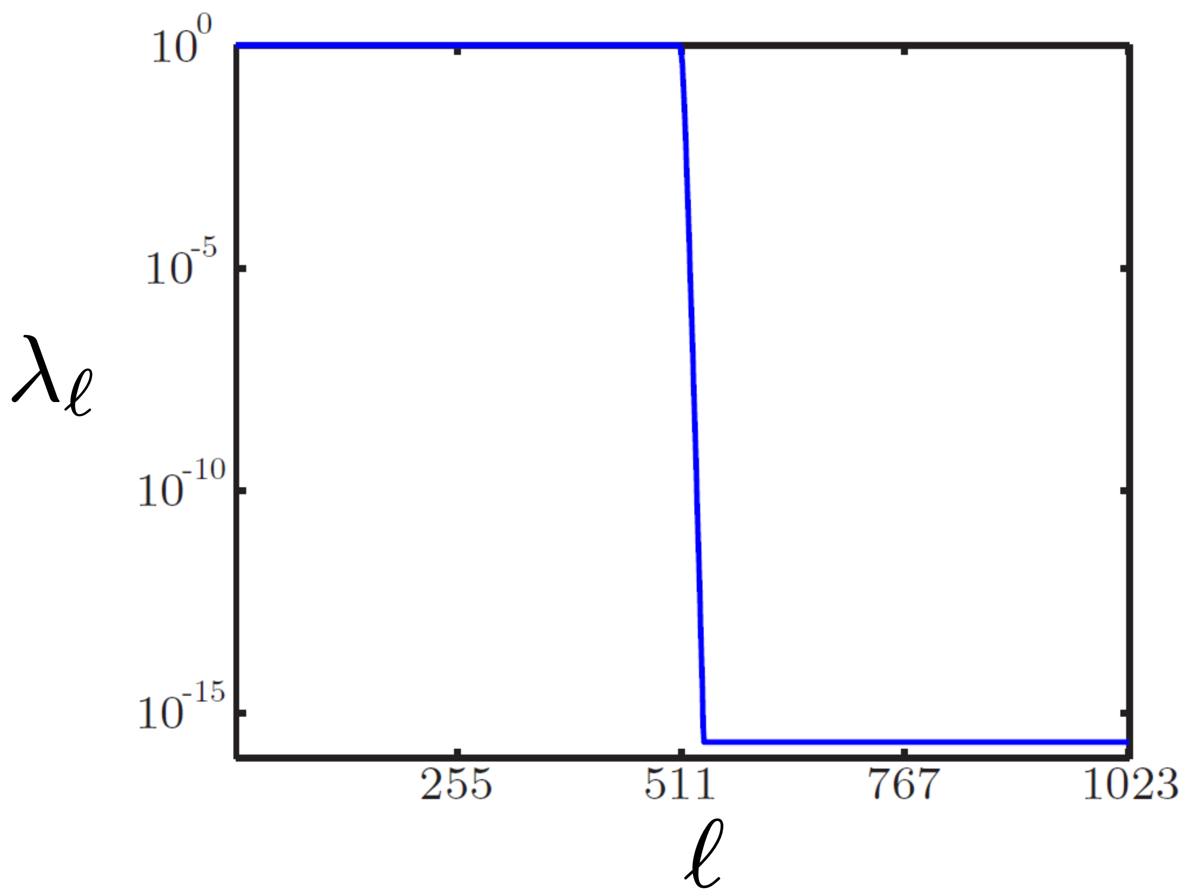


$N = 1024$

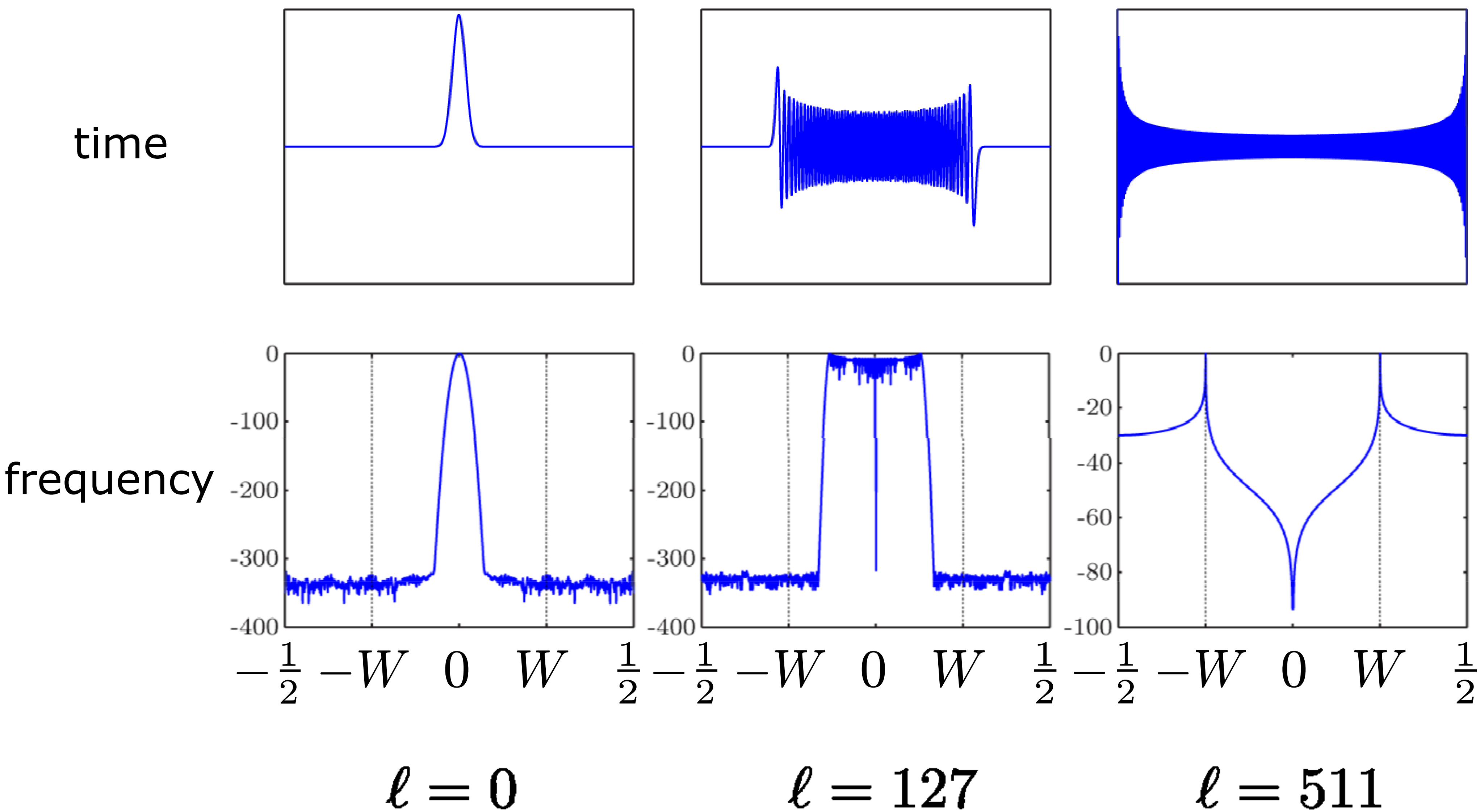
$W = \frac{1}{4}$

$2NW = 512$

The first $\approx 2NW$ eigenvalues ≈ 1.
The remaining eigenvalues ≈ 0.

DPSS Examples

$$N = 1024 \qquad W = \frac{1}{4}$$



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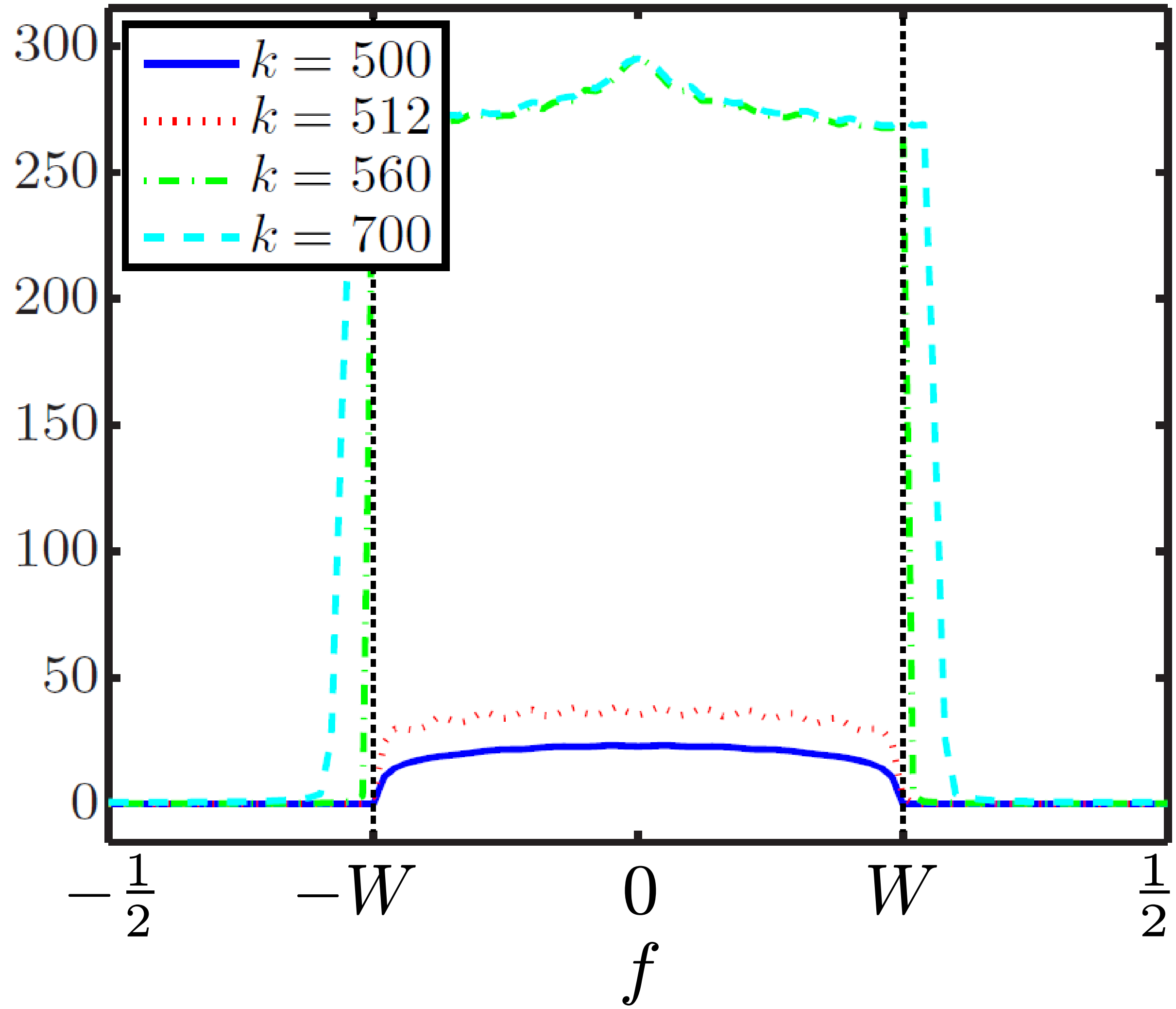
$$\int_{-W}^{W} \|\vec{e}_f - P_Q \vec{e}_f\|_2^2 \, df = \sum_{\ell=k}^{N-1} \lambda_\ell$$

Approximation of Bandlimited Signals

SNR (dB)

$$20\log_{10}\left(\frac{\|\vec{e}_f\|}{\|\vec{e}_f - P_Q\vec{e}_f\|}\right)$$



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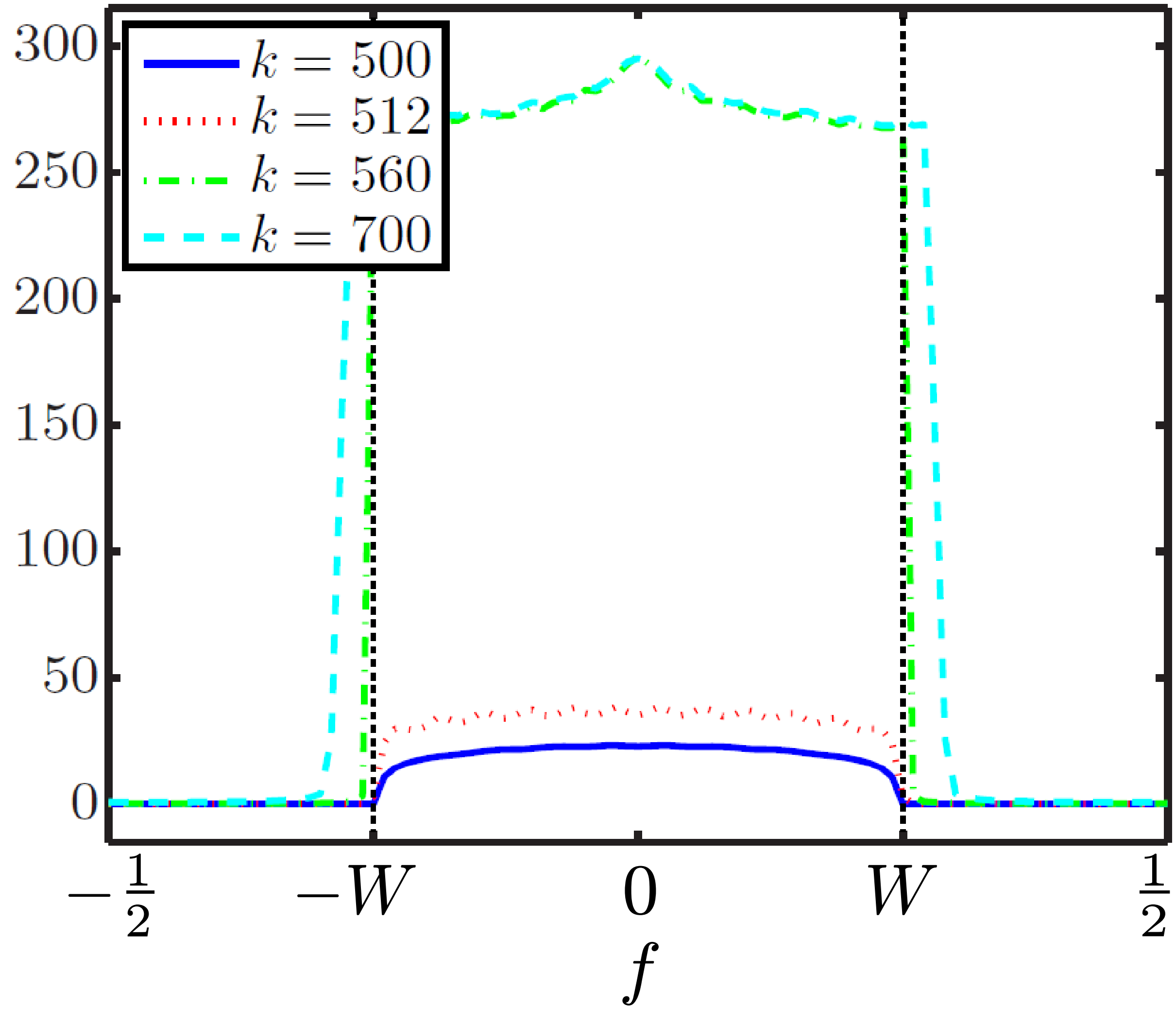
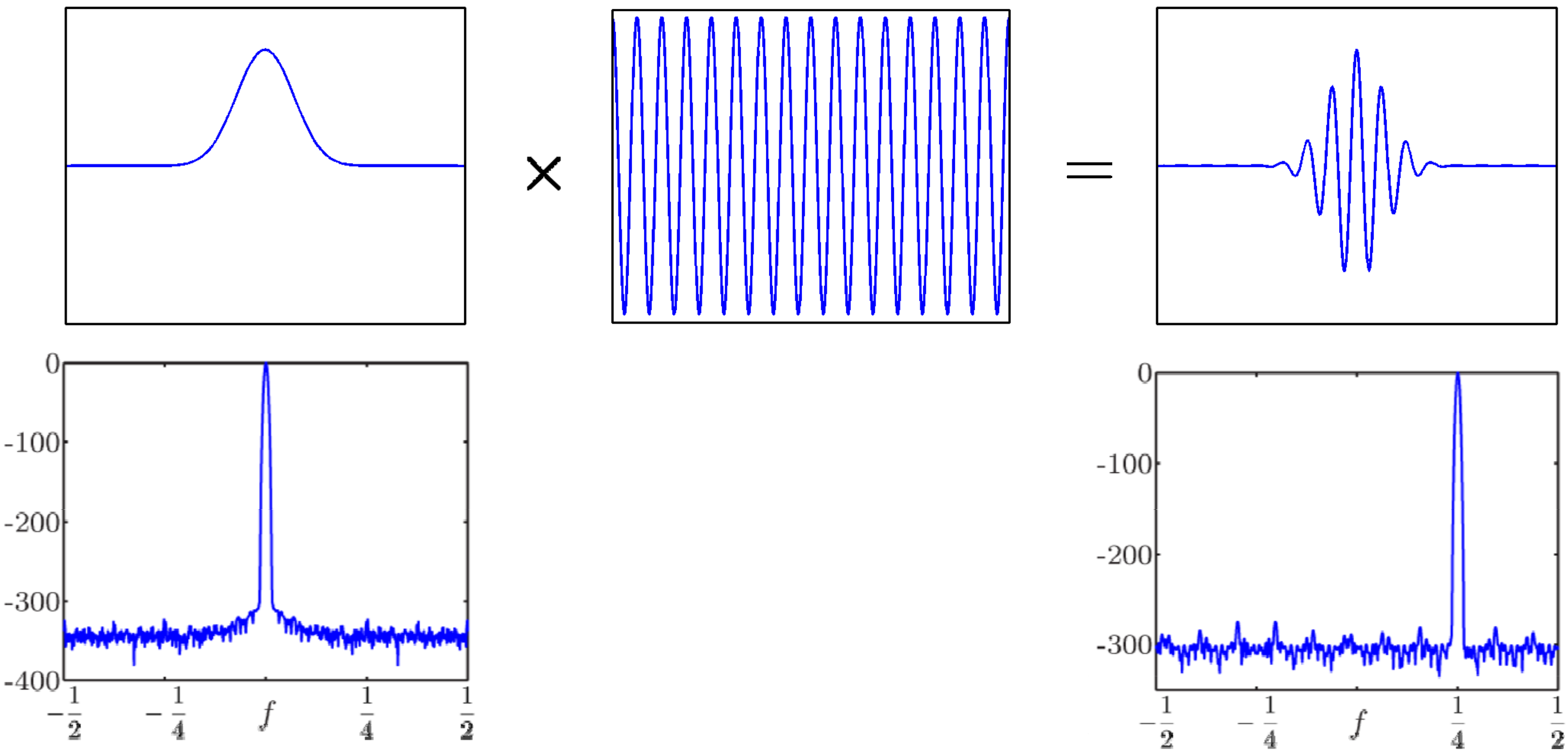


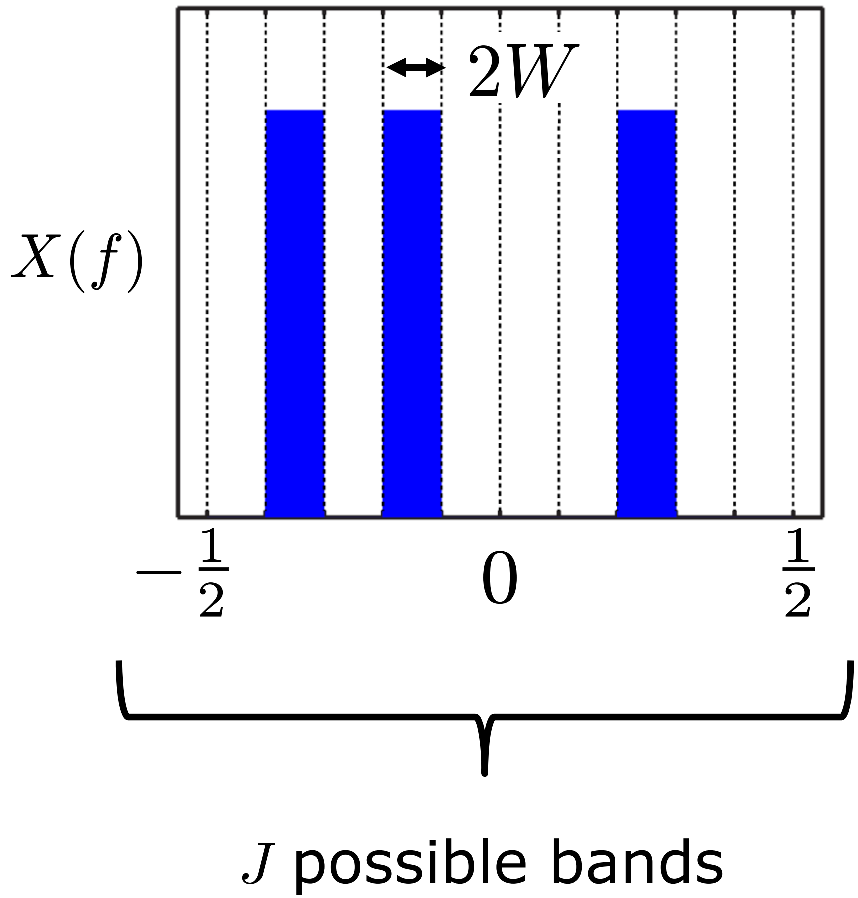
Most bandlimited analog signals,
when sampled and time-limited,
are well-approximated by the first k DPSS vectors.

DPSS's for Bandpass Signals



DPSS Dictionaries for CS

Modulate k DPSS vectors to center of each band:



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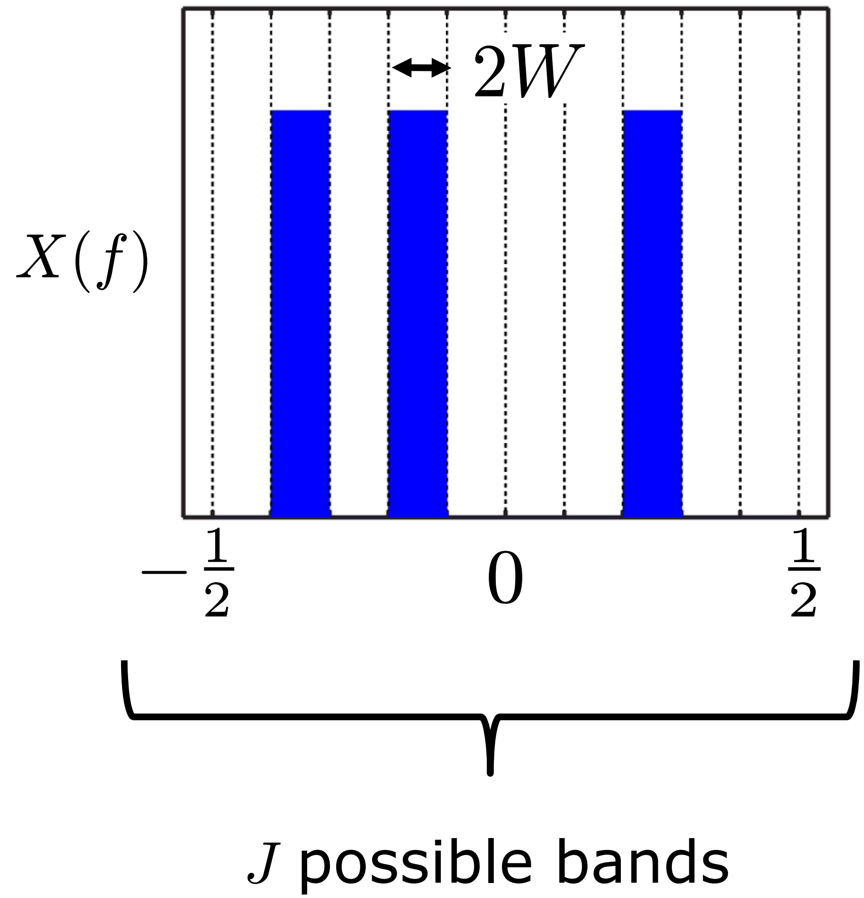
$$\Psi = [\Psi_1, \Psi_2, \ldots, \Psi_J]$$



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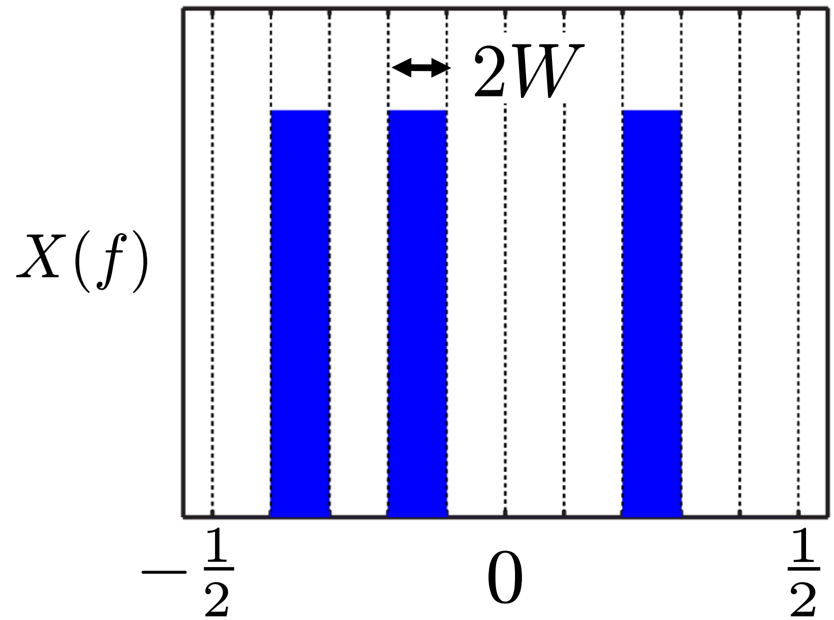

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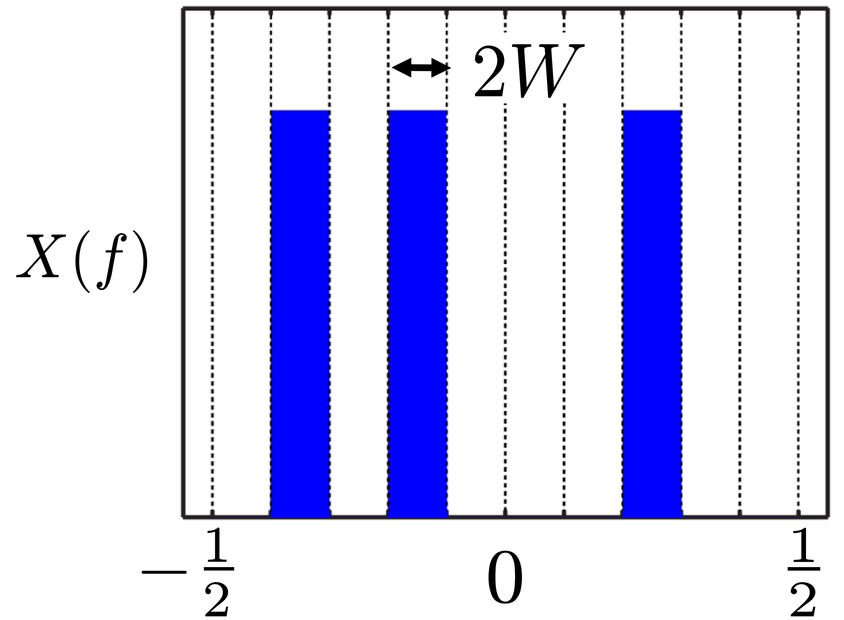

J possible bands

Most multiband analog signals,
when sampled and time-limited,
are well-approximated by a sparse representation in Ψ.

DPSS Dictionaries and the RIP

Theorem:
Suppose that Φ is sub-Gaussian and that the Ψ_i are constructed with $k = (1-\epsilon)2NW$. If

$$M \geq CS \log(N/S)$$

then with high probability $\Phi\Psi$ will satisfy the RIP of order S.

[W and Davenport, 2011]

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K occupied bands

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$$\frac{M}{N} \geq C' \frac{KB}{B_{\mathrm{nyq}}} \log\left(\frac{B_{\mathrm{nyq}}}{KB}\right)$$

[W and Davenport, 2011]

Block-Sparse Recovery

Nonzero coefficients of $\vec{\alpha}$ should be clustered in blocks according to the occupied frequency bands

$$\vec{x} = [\Psi_1, \Psi_2, \ldots, \Psi_J] \begin{bmatrix} \vec{\alpha}_1 \\ \vec{\alpha}_2 \\ \vdots \\ \vec{\alpha}_J \end{bmatrix}$$

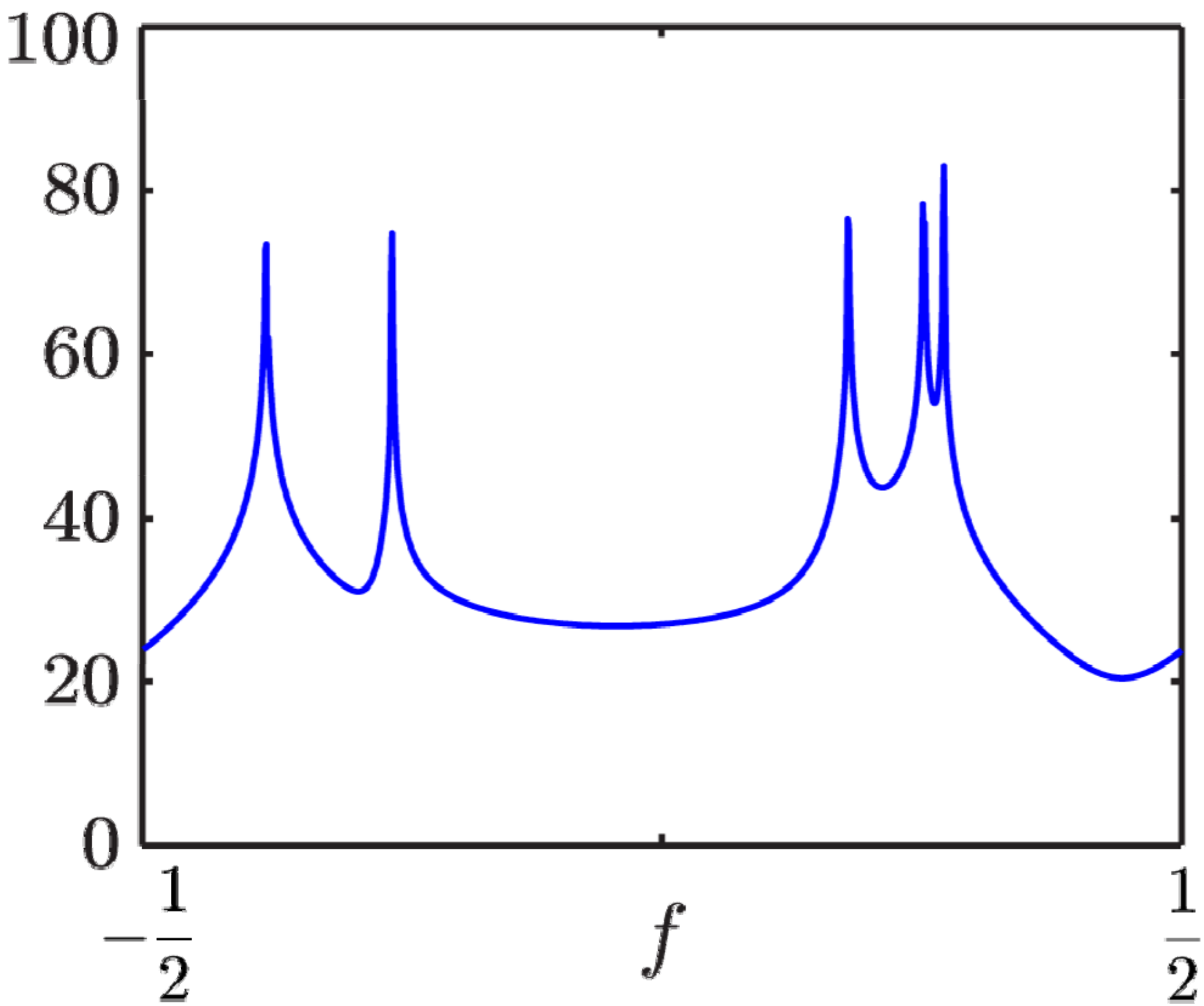
This can be leveraged to reduce the required number of measurements and improve performance through "model-based CS"

- –Baraniuk et al. [2008, 2009, 2010]
- –Blumensath and Davies [2009, 2011]

Recovery: DPSS vs DFT

$$\frac{B}{B_{\mathrm{nyq}}} = \frac{1}{512} \quad K = 5 \quad N = 1024 \quad S \approx 45$$

$$M = 128$$



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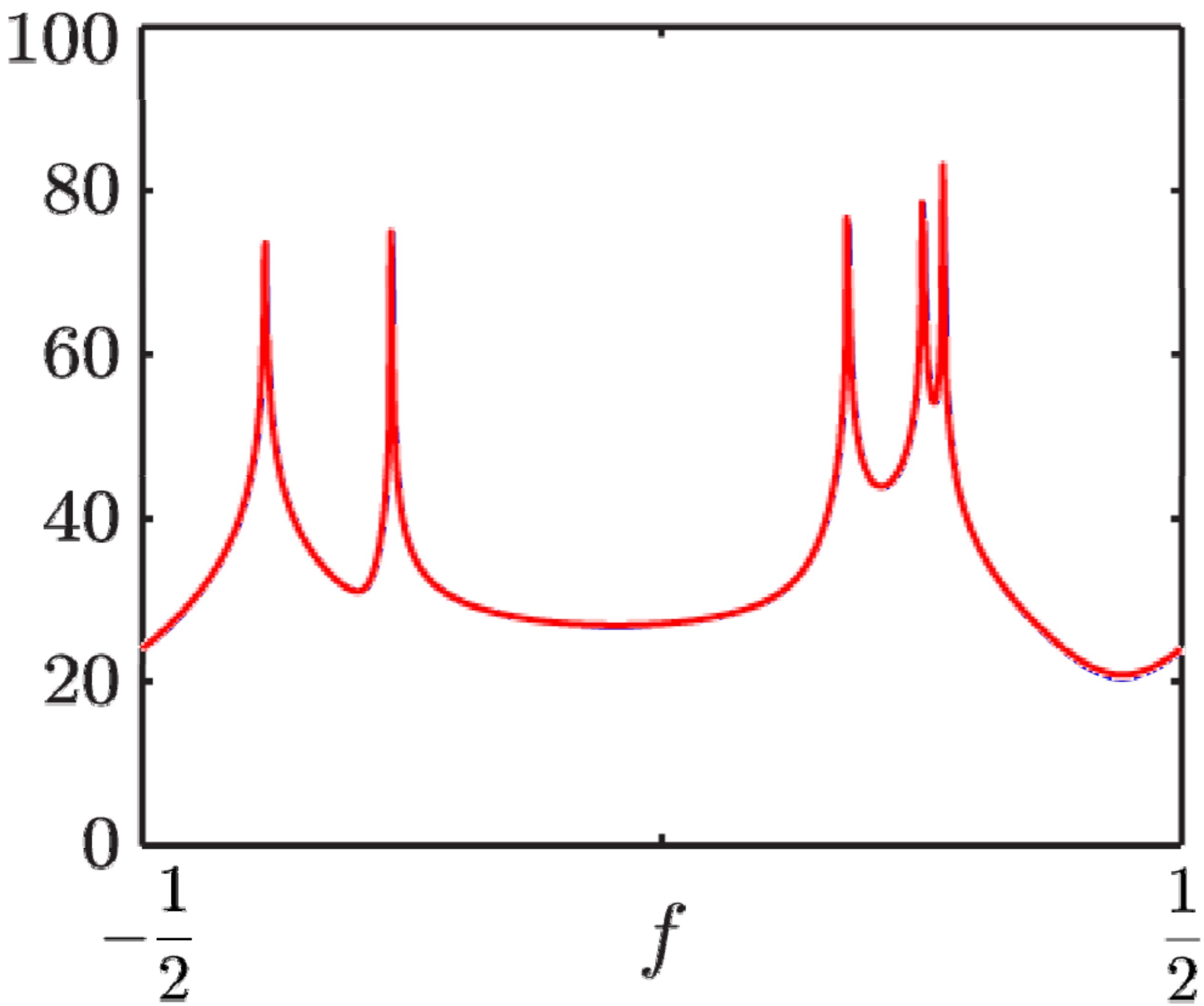

$\mathrm{DPSS}: \ \mathrm{SNR} = 54\mathrm{dB}$

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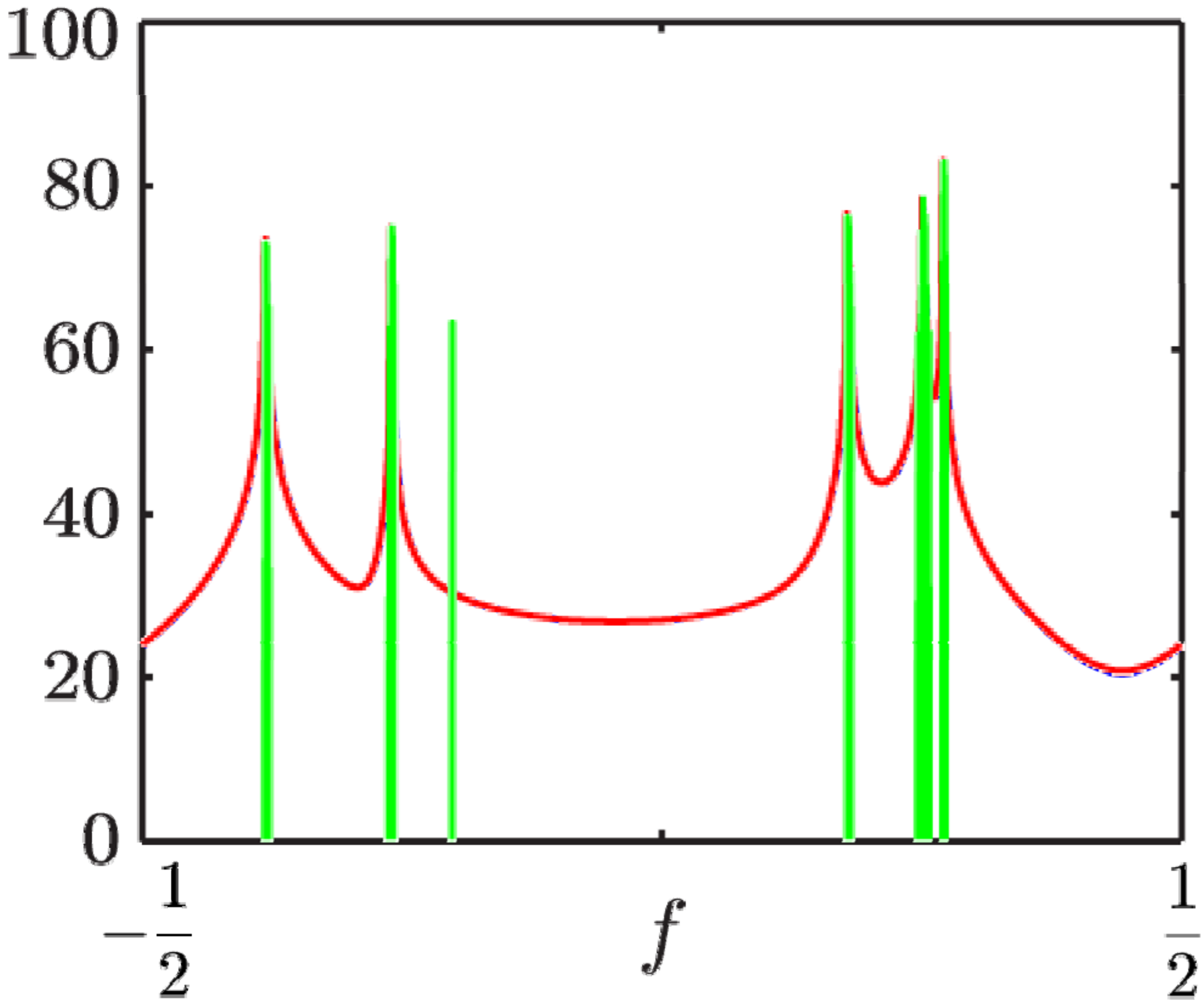
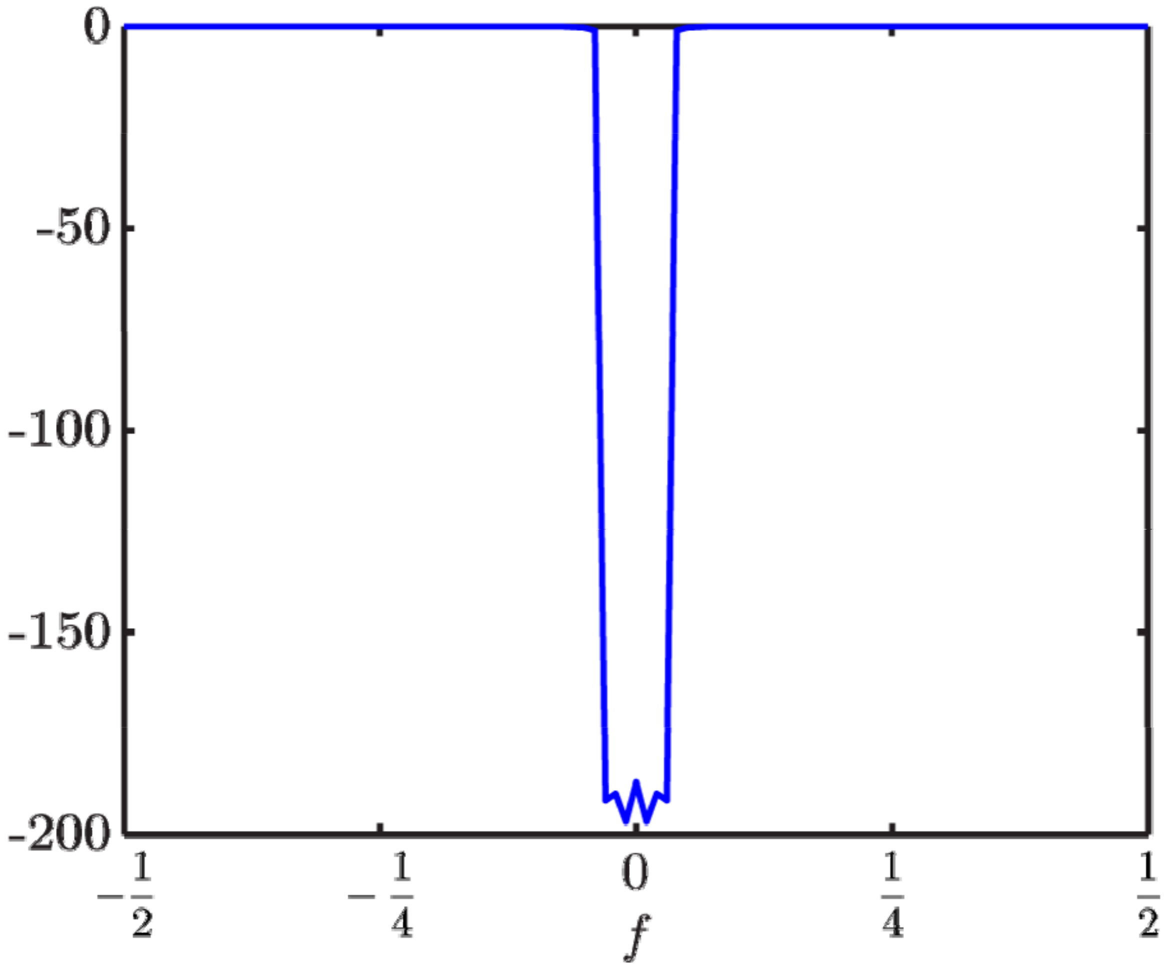


$\mathrm{DPSS}: \ \mathrm{SNR} = 54\mathrm{dB}$

$\mathrm{DFT}: \ \mathrm{SNR} = 12\mathrm{dB}$

Interference Cancellation

DPSS's can be used to cancel bandlimited interferers *without reconstruction.*



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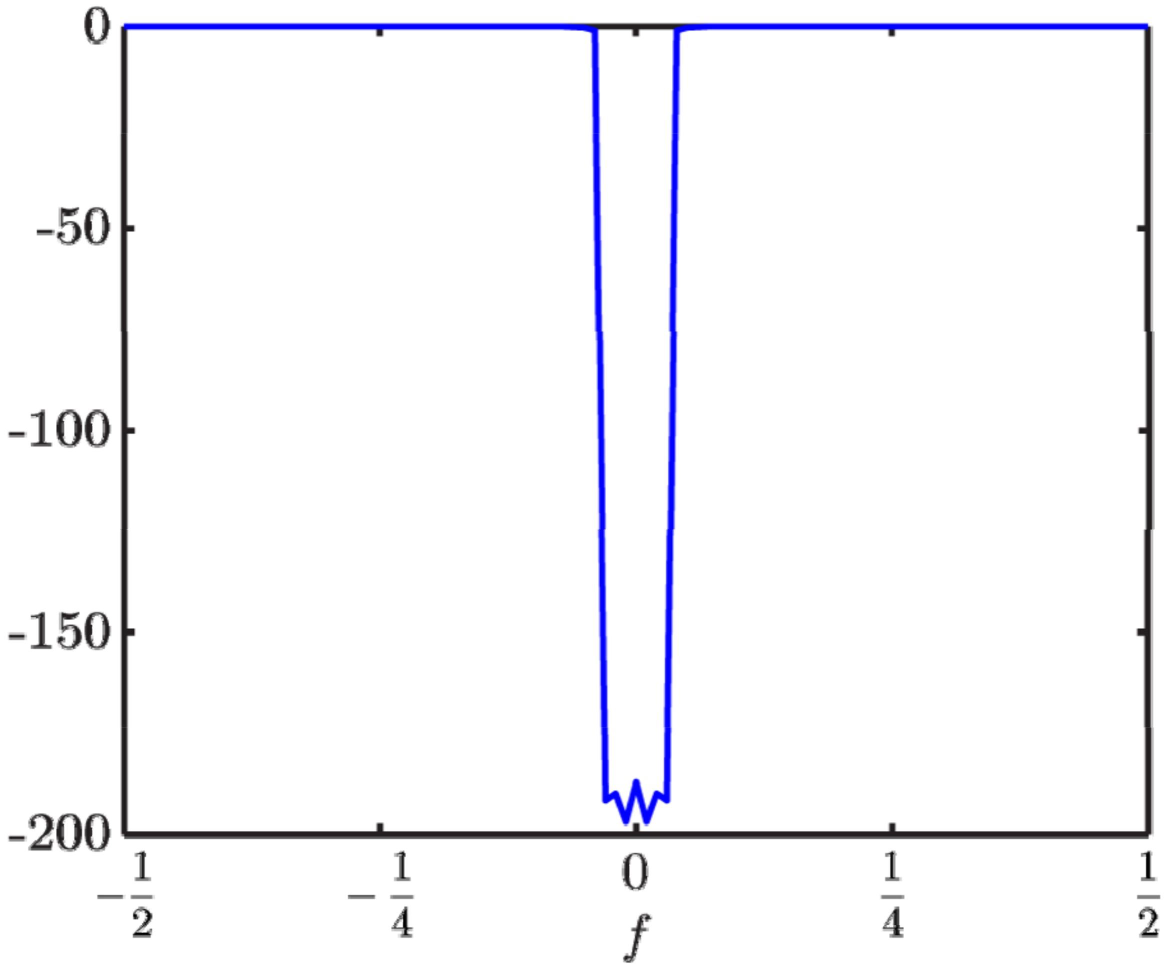

$$P = I - \Phi\Psi_i(\Phi\Psi_i)^\dagger$$

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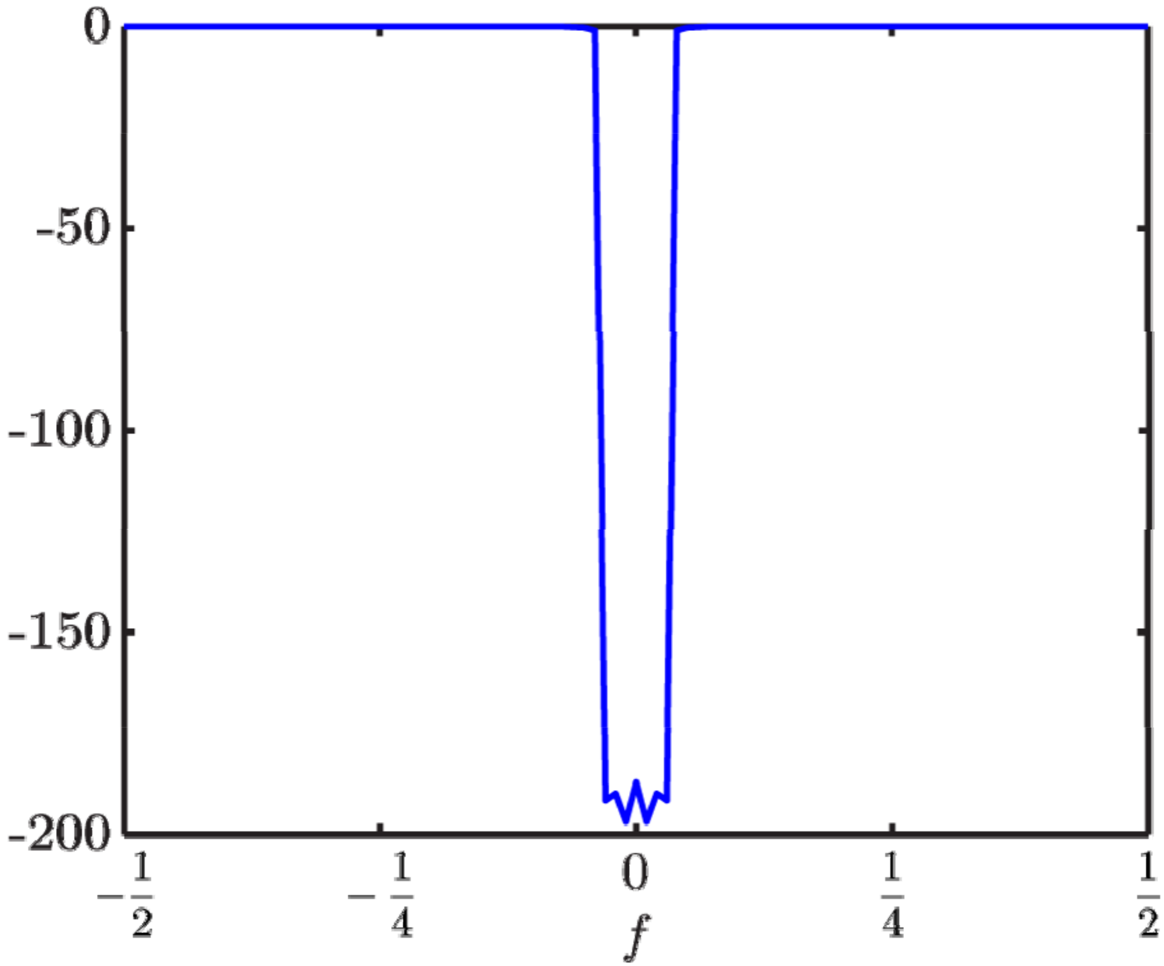

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Useful in *compressive signal processing* applications.

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  - reconstruction: zero for DPSS-sparse vectors
  - delicate balance in practice, but there is a sweet spot
- Related work
  - Gosse; Sejdić et al.; Senay et al.; Oh et al.; Izu and Lakey
  - none study <u>DPSS-based approximations</u> of <u>sampled multiband signals</u> and provide <u>CS recovery results</u>