An Efficient Dictionary for Reconstruction of Sampled Multiband Signals

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Two Regimes

Compressive Sensing (CS) is discrete-time, finite

\[ \vec{y} = \Phi \vec{x} \]
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Analog signals are continuous-time, infinite

\[ x(t) \rightarrow \text{random meas.} \rightarrow \ldots, y[-1], y[0], y[1], \ldots \]
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How compatible are these regimes?
Potential Challenges

Challenge 1:
Map analog sensing into matrix multiplication

Challenge 2:
Map analog sparsity into digital sparsity
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\[ y[m] = \langle \phi_m(t), x(t) \rangle \]
Challenge 1:
Map analog sensing into matrix multiplication

If $x(t)$ is bandlimited,

$$y[m] = \langle \phi_m(t), x(t) \rangle = \sum_{n=-\infty}^{\infty} x[n] \langle \phi_m(t), \text{sinc}(t/T_s - n) \rangle$$
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\[ \begin{array}{c}
\vec{y} \\
M \times 1 \\
\text{measurements} \\
\end{array} = \Phi \ \\
M \times N \\
\Phi \ \\
\vec{x} \\
N \times 1 \\
\text{Nyquist-rate samples of } x(t) \ \\
\end{array} \]
**Challenge 2:**

Map analog sparsity into digital sparsity

\[ \begin{align*}
\vec{x} & = \Psi \\
N \times 1 \text{ vector} & = \text{Nyquist-rate samples of } x(t) \\
\vec{\alpha} & \\
\end{align*} \]
## Candidate Models

<table>
<thead>
<tr>
<th>Model for $x(t)$</th>
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<th>Sparsity level for $\mathbf{x}$</th>
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<td>multitone, sum of $S$ “on-grid” tones</td>
<td>$\Psi = \text{DFT}$</td>
<td>$S$-sparse</td>
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The diagram below illustrates the frequency spectrum $X(F)$ of the signal $x(t)$, showing a sparse representation within the band $-\frac{B_{\text{nyq}}}{2}$ to $\frac{B_{\text{nyq}}}{2}$.
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<td>Multiband</td>
<td>$K$ occupied bands of bandwidth $B$</td>
<td>$\Psi = ?$</td>
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### Diagram

- Landau
- Bresler, Feng, Venkataramani
- Eldar, Mishali
The Problem with the DFT

\[ x(t) = \int_{-\frac{B}{2}}^{\frac{B}{2}} X(F) e^{j2\pi Ft} \, dF \]
The Problem with the DFT

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sampling

\[ x[n] = \int_{-W}^{W} X(f) e^{j2\pi fn} df, \forall n \]

\[ W = \frac{B}{2B_{\text{nyq}}} \]
The Problem with the DFT

\[ x[n] = \int_{-W}^{W} X(f) e^{j2\pi fn} df, \quad \forall n \]

\[ X(f) \]

\[ \text{DTFT} \]

\[ -\frac{1}{2} \quad 0 \quad \frac{1}{2} \]

\[ 2W \]
The Problem with the DFT

\[ x[n] = \int_{-W}^{W} X(f) e^{j2\pi fn} \, df, \quad \forall n \]

time-limiting

\[ \vec{x} = \sum_{k=0}^{N-1} X_k \vec{e}_k, \quad \vec{e}_f := \begin{bmatrix} e^{j2\pi f0} \\ e^{j2\pi f} \\ \vdots \\ e^{j2\pi f(N-1)} \end{bmatrix} \]
The Problem with the DFT

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NOT SPARSE
$$x[n] = \int_{-W}^{W} X(f)e^{j2\pi fn} \, df, \quad \forall n$$
Alternative Perspective

\[ x[n] = \int_{-W}^{W} X(f) e^{j2\pi fn} \, df, \quad \forall n \]

\[ T_N(x[n]) = \int_{-W}^{W} X(f) T_N(e^{j2\pi fn}) \, df, \quad \forall n \]
Building Blocks for Lowpass Signals

Time-limited complex exponentials form a “basis” for $\vec{x}$:

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$C^N$
Time-limited complex exponentials form a "basis" for $\vec{x}$:

$$\vec{x} = \int_{-W}^{W} X(f) \vec{e}_f \, df$$

$$\vec{e}_f := \begin{bmatrix} e^{j2\pi f_0} \\ e^{j2\pi f} \\ \vdots \\ e^{j2\pi f(N-1)} \end{bmatrix}$$

The problem: we need infinitely many of them.
Best Subspace Fit

Suppose that we wish to minimize

$$\int_{-W}^{W} \| \vec{e}_f - P_Q \vec{e}_f \|_2^2 \, df$$

over all subspaces $Q$ of dimension $k$. 
Best Subspace Fit

Suppose that we wish to minimize

$$\int_{-W}^{W} \|\vec{e}_f - P_Q\vec{e}_f\|_2^2 \, df$$

over all subspaces $Q$ of dimension $k$.

Optimal subspace is spanned by the first $k$ “DPSS vectors”.
Discrete Prolate Spheroidal Sequences (DPSS’s)

*Slepian [1978]:* Given an integer $N$ and $W \leq 0.5$, the DPSS’s are a collection of $N$ vectors

$$\vec{s}_0, \vec{s}_1, \ldots, \vec{s}_{N-1} \in \mathbb{R}^N$$

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The DPSS’s are perfectly time-limited, but when

$$\lambda_\ell \approx 1$$

they are highly concentrated in frequency.
DPSS Eigenvalue Concentration

$\lambda_\ell$

$N = 1024$

$W = \frac{1}{4}$
The first $\approx 2NW$ eigenvalues $\approx 1$. The remaining eigenvalues $\approx 0$. 

$N = 1024$
$W = \frac{1}{4}$

$2NW = 512$
DPSS Examples

\[ N = 1024 \quad W = \frac{1}{4} \]

\[ \ell = 0 \quad \ell = 127 \quad \ell = 511 \]
Recall: Best Subspace Fit

Suppose that we wish to minimize

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Recall: Best Subspace Fit

Suppose that we wish to minimize

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Optimal subspace is spanned by the first $k$ “DPSS vectors”.

$$\int_{-W}^{W} \left\| \tilde{e}_f - P_Q \tilde{e}_f \right\|_2^2 \, df = \sum_{\ell=k}^{N-1} \lambda_{\ell}$$
Approximation of Bandlimited Signals

SNR (dB)

$$20 \log_{10} \left( \frac{\| \tilde{e}_f \|}{\| \tilde{e}_f - P_Q \tilde{e}_f \|} \right)$$
Approximation of Bandlimited Signals

Most bandlimited analog signals, when sampled and time-limited, are well-approximated by the first $k$ DPSS vectors.
DPSS’s for Bandpass Signals
Modulate $k$ DPSS vectors to center of each band:

$X(f)$

$-\frac{1}{2}$ $0$ $\frac{1}{2}$

$J$ possible bands
Modulate \( k \) DPSS vectors to center of each band:

\[
\Psi = [\Psi_1, \Psi_2, \ldots, \Psi_J]
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approximately square if $k \approx 2NW$

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DPSS Dictionaries for CS

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approximately square if $k \approx 2NW$

Most multiband analog signals, when sampled and time-limited, are well-approximated by a sparse representation in $\Psi$. 

$X(f)$

$-\frac{1}{2}$ $0$ $\frac{1}{2}$

$2W$

$J$ possible bands
**Theorem:**
Suppose that $\Phi$ is sub-Gaussian and that the $\Psi_i$ are constructed with $k = (1 - \epsilon)2NW$. If

$$M \geq CS \log(N/S)$$

then with high probability $\Phi \Psi$ will satisfy the RIP of order $S$.

[W and Davenport, 2011]
DPSS Dictionaries and the RIP

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\( K \) occupied bands \( \rightarrow \) \( S \approx KNB / B_{nyq} \)
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$K$ occupied bands $\quad \Rightarrow \quad S \approx KNB/B_{\text{nyq}}$

$$\frac{M}{N} \geq C' \frac{KB}{B_{\text{nyq}}} \log \left( \frac{B_{\text{nyq}}}{KB} \right)$$

[W and Davenport, 2011]
Block-Sparse Recovery

Nonzero coefficients of $\tilde{\alpha}$ should be clustered in blocks according to the occupied frequency bands

$$\tilde{x} = [\Psi_1, \Psi_2, \ldots, \Psi_J] \begin{bmatrix} \tilde{\alpha}_1 \\ \tilde{\alpha}_2 \\ \vdots \\ \tilde{\alpha}_J \end{bmatrix}$$

This can be leveraged to reduce the required number of measurements and improve performance through “model-based CS”

– Baraniuk et al. [2008, 2009, 2010]
– Blumensath and Davies [2009, 2011]
Recovery: DPSS vs DFT

\[ \frac{B}{B_{nyq}} = \frac{1}{512} \quad K = 5 \quad N = 1024 \quad S \approx 45 \]

\[ M = 128 \]
Recovery: DPSS vs DFT

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DPSS: SNR = 54dB
Recovery: DPSS vs DFT

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\[ M = 128 \]

DPSS : SNR = 54dB

DFT : SNR = 12dB
Interference Cancellation

DPSS’s can be used to cancel bandlimited interferers without reconstruction.
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\[ P = I - \Phi \Psi_i (\Phi \Psi_i)\dagger \]
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Useful in compressive signal processing applications.
Summary

• DPSS’s can be used to efficiently represent most sampled multiband signals
  – far superior to DFT
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• Two types of error: *approximation* + *reconstruction*
  – approximation: small for most signals
  – reconstruction: zero for DPSS-sparse vectors
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  – far superior to DFT

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• Related work
  – Gosse; Sejdić et al.; Senay et al.; Oh et al.; Izu and Lakey
  – none study DPSS-based approximations of sampled multiband signals and provide CS recovery results