Operator-Theoretic Modeling for Radar in the Presence of Doppler

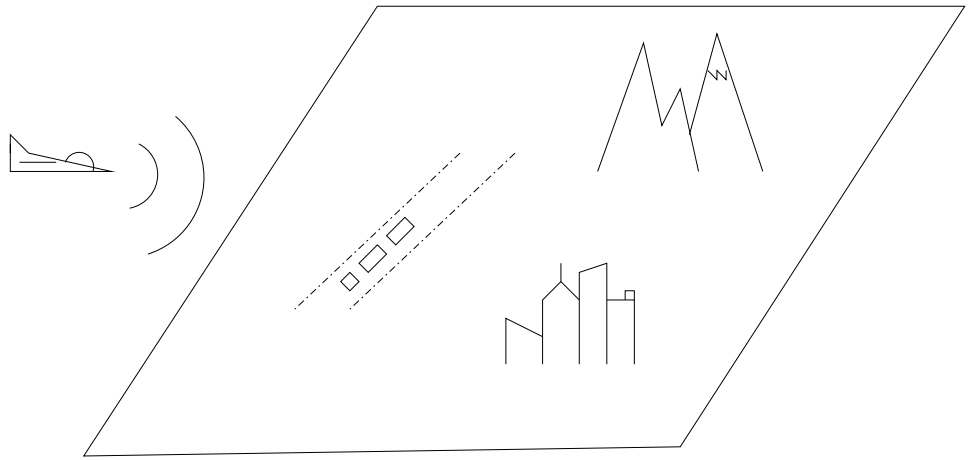
Doug Cochran[1], Stephen D. Howard[2], and Bill Moran[3]

Workshop on Sensing and Analysis of High-Dimensional Data

July 2011

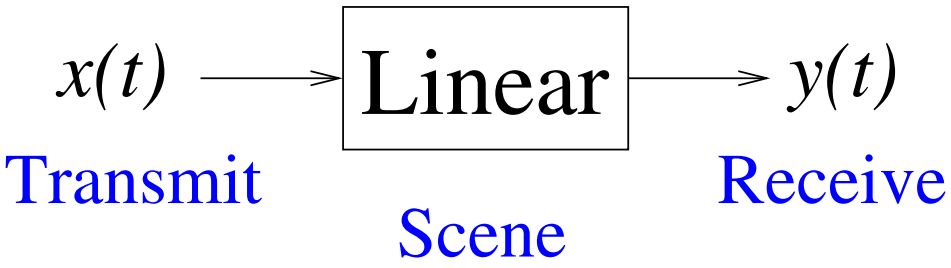
[1] Arizona State University, Tempe AZ, USA
[2] Defence Science & Technology Organisation, Edinburgh SA, Australia
[3] University of Melbourne, Parkville VIC, Australia

Optimal Waveform Design
Cochran Howard Moran
Introduction
H-S Model
Detection in AWGN
Detection in Clutter and Noise
Summary



- Components: transmitters, receivers, scene (targets and clutter), noise (low SNR)
- Under typical assumptions, range encoded as time delay and range rate encoded in frequency shift (narrowband approximation of Doppler)
- The radar chooses the transmitted waveforms and beamforms and the corresponding receiver processing

Optimal Waveform Design
Cochran Howard Moran
Introduction
H-S Model
Detection in AWGN
Detection in Clutter and Noise
Summary



- In a SISO radar, the scene acts on the transmitted waveform and produces the received waveform
- For typical scenarios, this transformation is well modeled as a linear operator on $L^2(\mathbb{R})$ plus noise; target and clutter operators summed
- The design objective varies with application goal (detection, classification, scene characterization)

Introduction
Willard Miller

Optimal Waveform Design

Cochran Howard Moran

Introduction

H-S Model

Detection in AWGN

Detection in Clutter and Noise

Summary

Introduction
Operator view of radar

Optimal Waveform Design

Cochran Howard Moran

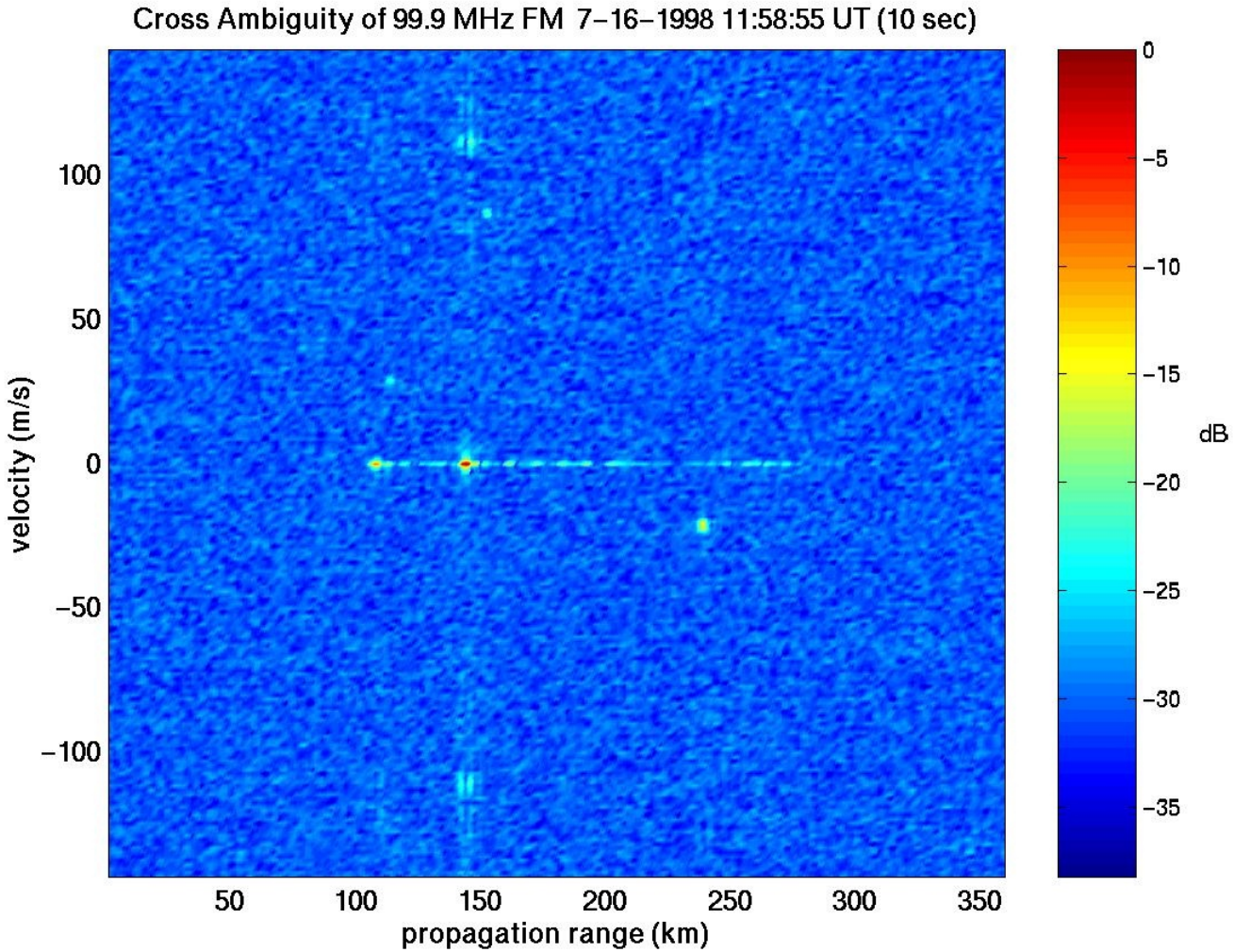
Introduction

H-S Model

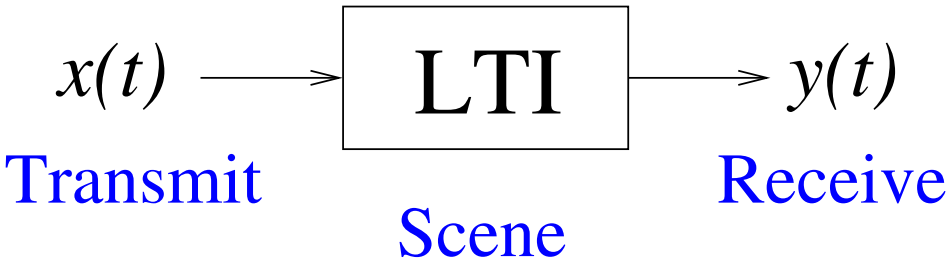
Detection in AWGN

Detection in Clutter and Noise

Summary



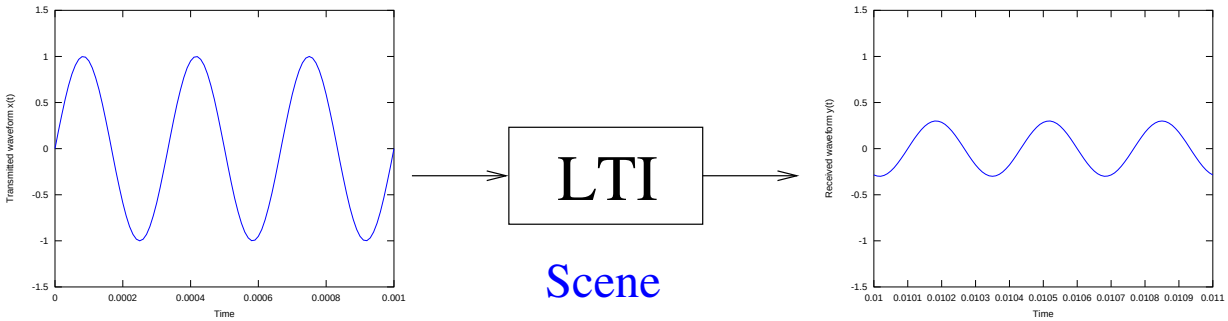
Optimal Waveform Design
Cochran Howard Moran
Introduction
H-S Model
Detection in AWGN
Detection in Clutter and Noise
Summary



- Modeling of a radar scene as a linear time-invariant (LTI) system is common in the radar literature
- In particular, LTI models have received attention in connection with waveform design
- But LTI models have a fundamental limitation...

Introduction
Limitations of LTI models

Optimal Waveform Design
Cochran Howard Moran
Introduction
H-S Model
Detection in AWGN
Detection in Clutter and Noise
Summary



- Sinusoids are eigenfunctions of LTI systems
- So the only frequency components present in the output are those present in the input
- **LTI models cannot accommodate Doppler!**

Hilbert-Schmidt Operator Model

Hilbert-Schmidt Operators

Our proposal: Replace LTI operator scene models with Hilbert-Schmidt operator scene models, which accommodate Doppler and have other desirable features

- Fundamental building block: *displacement operator* $D(p,q) : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ by

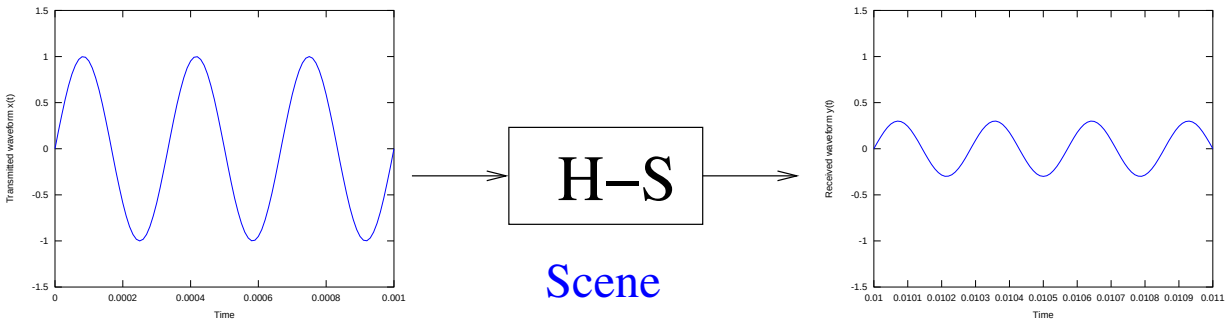
$$D(p,q)f(t) = e^{ipt} f(t-q)$$

  Here p denotes frequency shift and q time shift; the (p,q) notation is adopted from physics where they are standard for phase space coordinates
- Every Hilbert-Schmidt class operator $S : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ can be written as a superposition of displacement operators

$$S = \int_p \int_q s(p,q) D(p,q) \, dp \, dq$$

Hilbert-Schmidt Operator Model
Properties of Hilbert-Schmidt Operators

Optimal Waveform Design
Cochran Howard Moran
Introduction
H-S Model
Detection in AWGN
Detection in Clutter and Noise
Summary



- H-S operators can impart frequency shifts as well as time and phase shifts
- So HS models **can** accommodate Doppler
- LTI (convolutional) operators are a subclass of the H-S class
- But...

$$D(p,q)D(p',q') = D(p',q')D(p,q)e^{i(pq'-p'q)}$$

i.e., H-S operators are generally non-commutative

Hilbert-Schmidt Operator Model
Scene Model

- *Discrete target with position* (p_0, q_0)*:*

$$\sum_{(p,q)\in N(0,0)} s_t(p,q)D(p,q) \quad D(p_0,q_0)$$

- *Clutter:*

$$\sum_{(p,q)} s_c(p,q)D(p,q)$$

  e.g., Markov random field, homogeneous, Gaussian...
- Both target and clutter are often sparse, but not in a finite library
- *Noise:* Additive Gaussian $n(t)$

Optimal Waveform Design
Cochran Howard Moran
Introduction
H-S Model
Detection in AWGN
Detection in Clutter and Noise
Summary

Standard radar set-up...

- *Transmitter:* Baseband waveform $w(t)$ with modulation to carrier frequency p_0
- *Receiver:* Demodulation plus signal processing (H)

Typical goal:
Define optimal systems (waveform w & processing H) for various classes of problems, ultimately with *constraints* (e.g., power, bandwidth, time) on w

Optimal Waveform Design
Cochran Howard Moran
Introduction
H-S Model
Detection in AWGN
Detection in Clutter and Noise
Summary

Receiver data at baseband...

- *Target present* (H_1)

$$x(t) = \sum_{(p,q)\in N(0,0)} s_t(p,q) D(p_0,q_0)^\dagger \cdot D(p,q) D(p_0,q_0) w(t) + n(t)$$

$$= Bw(t) + n(t) \sim \mathcal{N}(Bw(t), \sigma^2)$$

where $D(p_0,q_0)^\dagger = \exp(ip_0q_0)D(-p_0,-q_0)$ is the adjoint of $D(p_0,q_0)$

- *Target absent* (H_0)

$$x(t) = n(t) \sim \mathcal{N}(0, \sigma^2)$$

Basic Detection Problem
Waveform optimization

- Define

$$B = \sum_{(p,q)\in N(0,0)} s_t(p,q) D(p_0,q_0)^\dagger D(p,q) D(p_0,q_0)$$

- Detection is optimized for fixed $||w||$ if $||Bw||^2$ is maximized; i.e., w should maximize

$$\langle Bw, Bw \rangle = \left\langle w, B^\dagger B w \right\rangle$$

  and should thus be an eigenfunction of the non-negative definite operator $B^\dagger B$ corresponding to its maximal eigenvalue
- If w is constrained by power rather than energy, the solution changes (another day...)

Basic Detection Problem
Scene Model With Clutter

- Signal at the receiver

$$s(t) = Tw(t) + Cw(t) + n(t)$$

  - T is a H-S class target operator
  - C is a H-S class clutter operator
  - n is white Gaussian noise
- Receiver processes s with a H-S operator H
- Detection decision on the basis of the statistic[1]

$$r(t_0) = Hs(t_0) = (HTw + HCw + Hn)(t_0)$$

[1] See S. U. Pillai et al., "Waveform design optimization using bandwidth and energy considerations," *Proceedings of the IEEE Radar Conference*, pp. 1–5, May 2008.

Basic Detection Problem
SINR

- The associated signal to interference plus noise ratio (SINR) is

$$\text{SINR}(t_0) = \frac{|HTw(t_0)|^2}{\text{E}|HCw(t_0) + Hn(t_0)|^2} \tag{1}$$

- The initial objective is to maximize $\text{SINR}(t_0)$ for a given waveform w
  - The problem of optimizing over w comes up later

Basic Detection Problem
Noise Term

- H may be represented as an integral operator with kernel $\Phi : \mathbb{R}^2 \to \mathbb{C}$

$$Hf(t) = \int_{\mathbb{R}} \Phi(t, \tau) f(\tau)\, d\tau$$

- By direct calculation,

$$\mathsf{E}|Hn(t)|^2 = \sigma^2 \int |\Phi(t, \tau)|^2\, d\tau$$

- Denoting $h_{t_0} = \Phi(t_0, \cdot)$, this gives

$$\mathsf{E}|Hn(t_0)|^2 = \sigma^2 ||h_{t_0}||^2 \tag{2}$$

Basic Detection Problem
Clutter Term

- For the clutter term,

$$\begin{aligned} |HCw(t_0)|^2 &= \int\int h_{t_0}(\tau)\overline{h_{t_0}(u)}Cw(\tau)\overline{Cw(u)}\,d\tau\,du \\ &= \langle h_{t_0}, G_C h_{t_0}\rangle \end{aligned} \tag{3}$$

  - $G_C \geq 0$ is a "waveform dependent" Hermitian operator defined by

$$G_C f(t) = \int_{\mathbb{R}} \mathsf{E}[Cw(t)\overline{Cw(\tau)}]\, f(\tau)\, d\tau$$

  - Note that the clutter operator may be regarded as random; no issues if n has zero mean and is independent of C
- For deterministic (known) clutter

$$G_C = CP_wC^\dagger$$

where P_w is projection onto span w

Basic Detection Problem
Signal Term and Maximization Formulation

Optimal Waveform Design
Cochran Howard Moran
Introduction
H-S Model
Detection in AWGN
Detection in Clutter and Noise
Summary

- In inner product form, the SINR numerator is

$$|HTw(t_0)|^2 = \langle h_{t_0}, P_{Tw} h_{t_0} \rangle \tag{4}$$

  - P_{Tw} is the rank-one projection operator onto span Tw
- Substituting (2), (3), and (4) into (1) yields

$$\text{SINR}(t_0) = \frac{\langle h_{t_0}, P_{Tw} h_{t_0} \rangle}{\langle h_{t_0}, (\sigma^2 \mathbb{I} + G_C) h_{t_0} \rangle} \tag{5}$$

- SINR maximization is thus a generalized eigenvalue problem
  - $P_{Tw} > 0$ with rank one
  - $G_C \geq 0$, so $(\sigma^2 \mathbb{I} + G_C) > 0$ provided $\sigma^2 > 0$
- Hence SINR is maximized by setting

$$h_{t_0} = h_{\max} = (\sigma^2 \mathbb{I} + G_C)^{-1/2} Tw$$

Basic Detection Problem
Maximizing SINR

- Denote $G = (\sigma^2 \mathbb{I} + G_C) > 0$ and observe G is self-adjoint
- Then $h_{\max} = G^{-1/2} T w$

$$\text{SINR}_{\max} = \frac{\left\langle G^{-1/2} T w, P_{Tw} G^{-1/2} T w \right\rangle}{\left\langle G^{-1/2} T w, G^{1/2} T w \right\rangle}$$

- With $f = Tw$,

$$\begin{aligned} \text{SINR}_{\max} &= \frac{\left\langle G^{-1/2} f, P_{Tw} G^{-1/2} f \right\rangle}{\left\langle G^{-1/2} f, G^{1/2} f \right\rangle} \\ &= \frac{\left\langle f, G^{-1/2} P_{Tw} G^{-1/2} f \right\rangle}{\left\langle f, f \right\rangle} \end{aligned}$$

Basic Detection Problem
Maximizing SINR (cont'd)

- $\text{SINR}_{\max}$ is the maximal eigenvalue of $G^{-1/2} P_{Tw} G^{-1/2} \geq 0$
  - Rank-one $\Longrightarrow$ its trace is its one positive eigenvalue
- Hence

$$\begin{aligned} \text{SINR}_{\max} &= \text{Tr}(G^{-1/2} P_{Tw} G^{-1/2}) \\ &= \left\langle G^{-1/2} f, G^{-1/2} f \right\rangle \\ &= \left\langle f, G^{-1} f \right\rangle = \left\langle Tw, G^{-1} Tw \right\rangle \\ &= \left\langle Tw, (\sigma^2 \mathbb{I} + G_C)^{-1} Tw \right\rangle \end{aligned}$$

- This reduces to past results for LTI scene models

Basic Detection Problem
Waveform Optimization

- With deterministic clutter operator C, the problem of finding a waveform $w_{\max}$ to maximize the SINR is

$$w_{\max} = \arg\max \left\langle Tw, (\sigma^2 \mathbb{I} + C P_w C^\dagger)^{-1} Tw \right\rangle$$

  - The presence of P_w in the denominator makes this problem "non-linear"
- We are investigating an iterative algorithm
  1. Take w_0 to be an eigenvector corresponding to the largest eigenvalue of $T^\dagger T$ ("target-driven")
  2. For $k > 0$, solve the "linear" problem

$$w_1 = \arg\max \left\langle Tw, (\sigma^2 \mathbb{I} + C P_{w_{k-1}} C^\dagger)^{-1} Tw \right\rangle$$

- "Clutter-driven" perspective leads to the same idea

Determining the Scene

Can we get the scene operator from a few measurements?

- The preceding examples assume the scene operator is known or hypothetical scene operators are being tested
- Scene operators are often composed of a sparse superposition of $D(p, q)$ operators
- Radar measures the operator one dimension at a time; time is often limited
- Sparsity assumptions of radar scenes may be helpful in regularizing this problem, but there are issues...
  - Operators are infinite-dimensional
  - Sparsity is not in a finite set, and is not usually well approximated this way
  - The SNR is usually *very* low

Summary
Summary and Ongoing Work

Optimal Waveform Design

Cochran Howard Moran

Introduction

H-S Model

Detection in AWGN

Detection in Clutter and Noise

Summary

- Hilbert-Schmidt class operators may be suitable to model radar scenes with Doppler
- Standard detection and waveform optimization problems can be formulated and solved in this framework
- The problem of determining such operators from small numbers of rank-one measurements has not been treated, at lease in the radar literature
- Sparsity assumptions of radar scenes may be helpful in regularizing this problem