Wavelets and Filter Banks on Graphs

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Joint work with David Shuman

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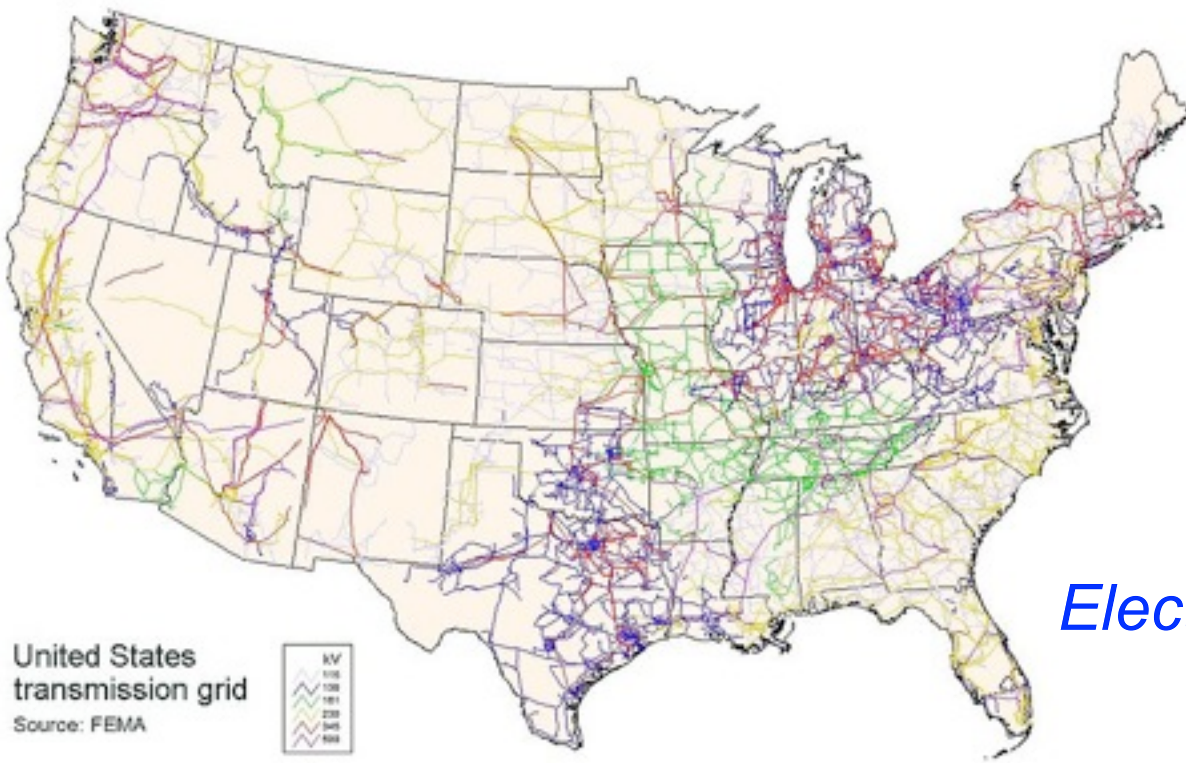
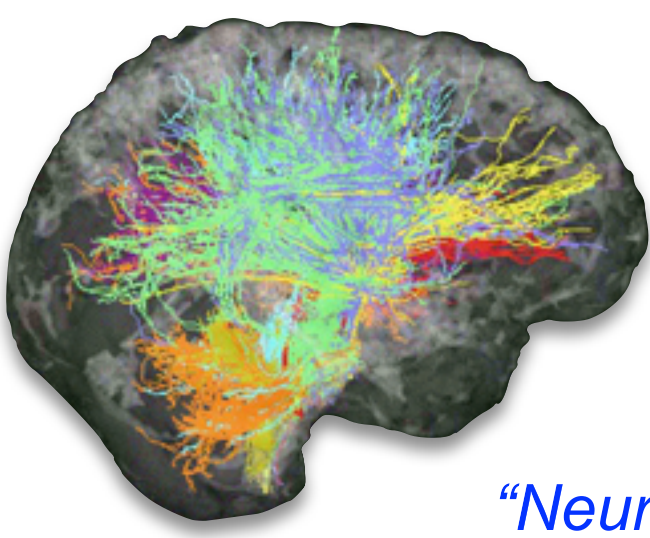
Processing Signals on Graphs

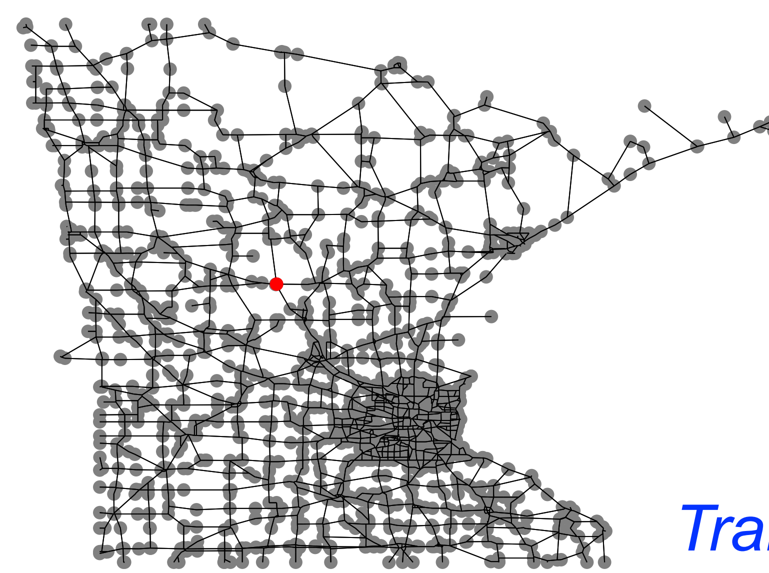
Social Network



Electrical Network

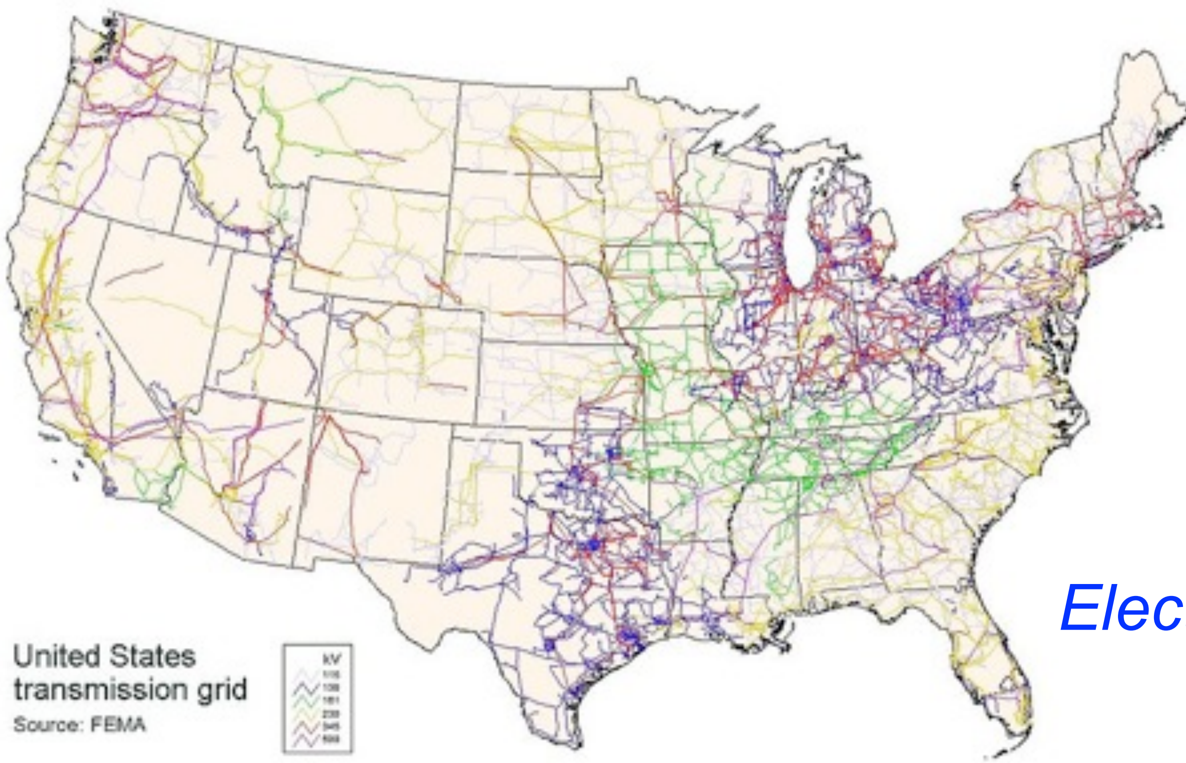


“Neuronal” Network



Transportation Network

Processing Signals on Graphs



Electrical Network

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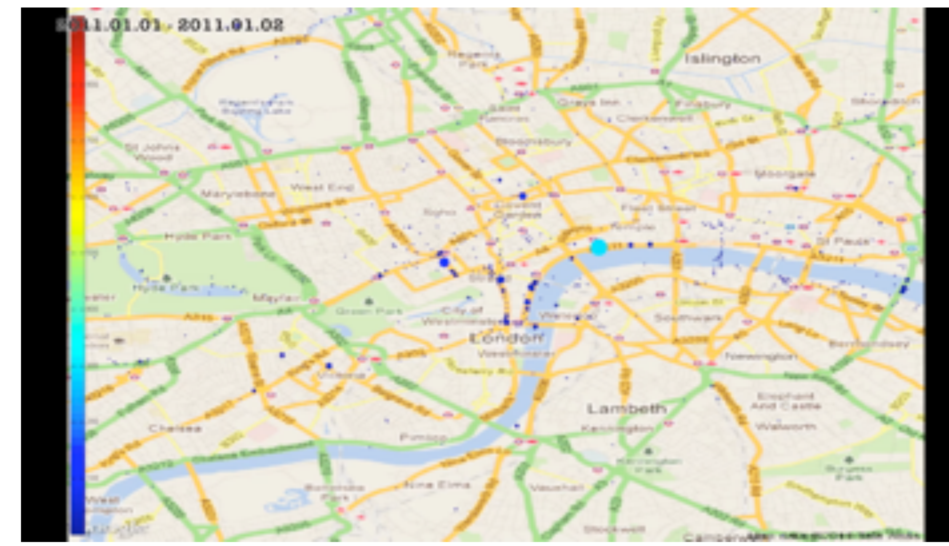

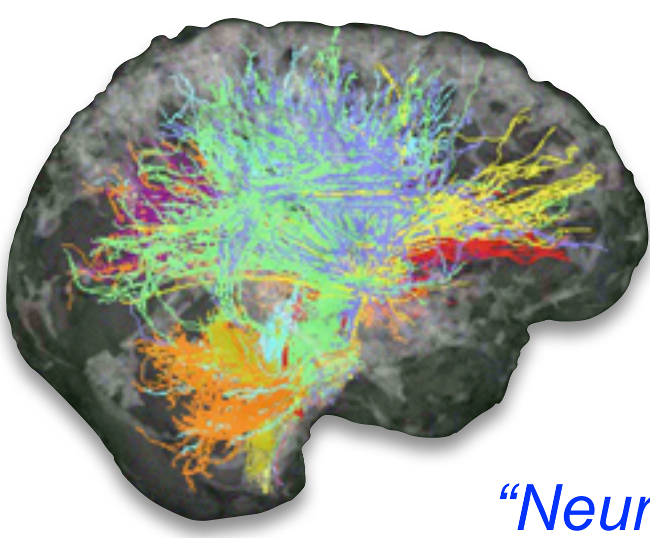

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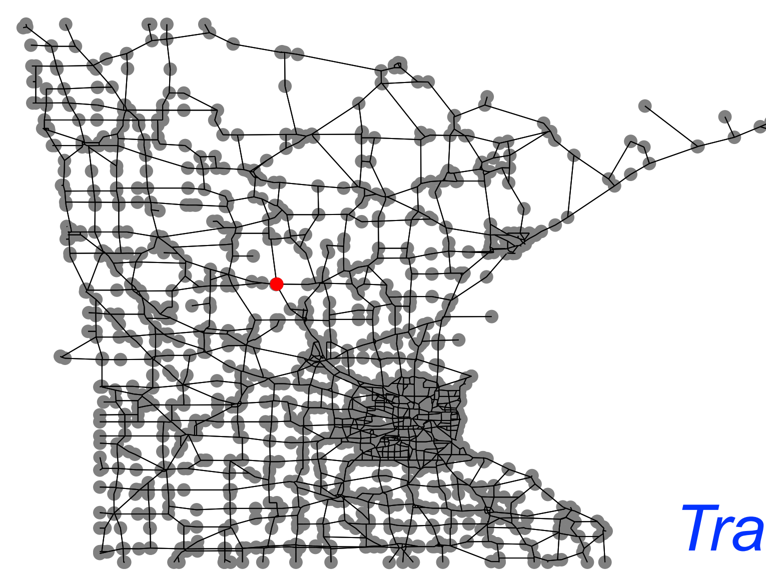

Transportation Network

Short outline

- Summary of one wavelet construction on graphs
  - multiscale, filtering
- Pyramidal algorithms
  - polyphase components and downsampling
  - the Laplacian Pyramid
  - 2-channels, critically sampled filter banks ?

Spectral Graph Wavelets

$G\!=\!(E,V)$ a weighted undirected graph, with Laplacian $\mathcal{L} = D - A$

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Translation (localization):

Define $\psi_{t,j} = T_g^t \delta_j$ response to a delta at vertex j

$$\psi_{t,j}(i) = \sum_{\ell=0}^{N-1} g(t\lambda_\ell)\phi_\ell^*(j)\phi_\ell(i) \qquad \mathcal{L}\phi_\ell(j) = \lambda_\ell\phi_\ell(j)$$

$$\psi_{t,a}(u) = \int_{\mathbb{R}} d\omega\, \hat{\psi}(t\omega)e^{-j\omega a}e^{j\omega u}$$

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And so formally define the graph wavelet coefficients of f:

$$W_f(t,j) = \langle \psi_{t,j}, f \rangle \qquad W_f(t,j) = T_g^t f(j) = \sum_{\ell=0}^{N-1} g(t\lambda_\ell)\hat{f}(\ell)\phi_\ell(j)$$

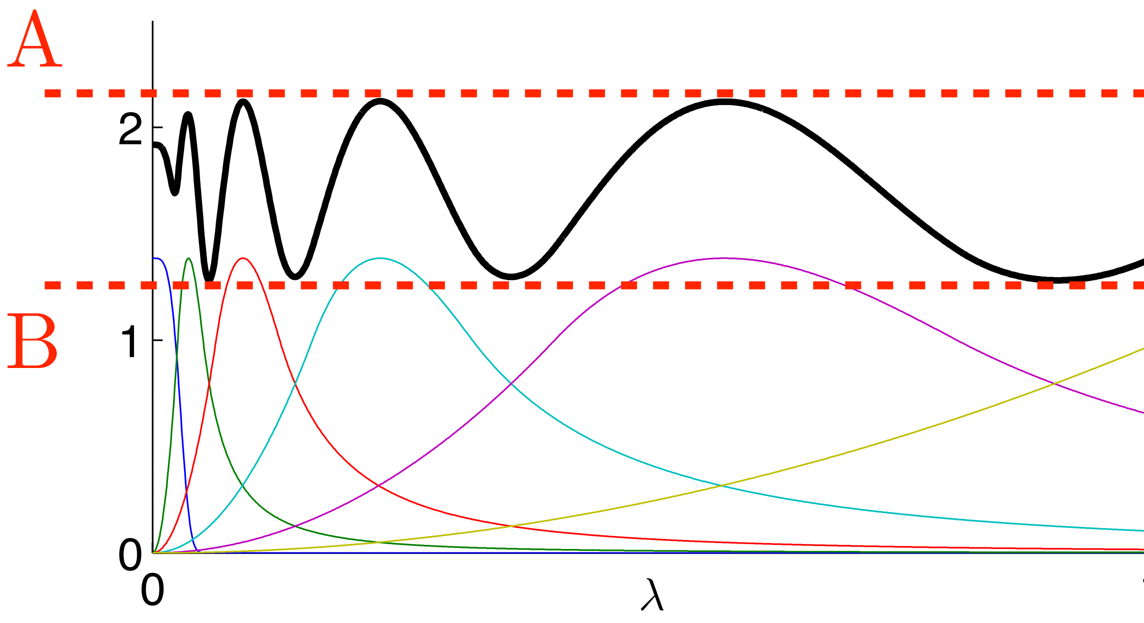
Frames

$\exists\, A, B > O,\ \exists h : \mathbb{R}_+ \to \mathbb{R}_+$ (i.e. scaling function)

$$0 < A \leqslant h^2(u) + \sum_s g(t_s u)^2 \leqslant B < \infty$$

scaling function   *wavelets*

$\phi_n = T_h \delta_n = h(\mathcal{L})\delta_n$

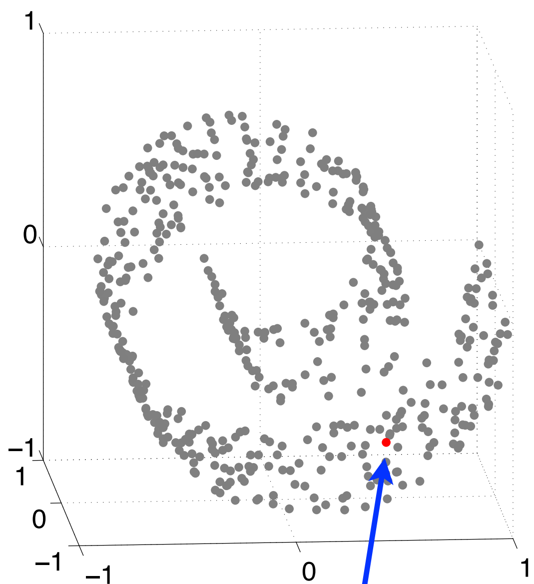


<u>A simple way to get a tight frame:</u>

$$\gamma(\lambda_\ell) = \int_{1/2}^{1} \frac{dt}{t}\, g^2(t\lambda_\ell) \quad \Longrightarrow \quad \tilde{g}(\lambda_\ell) = \sqrt{\gamma(\lambda_\ell) - \gamma(2\lambda_\ell)}$$

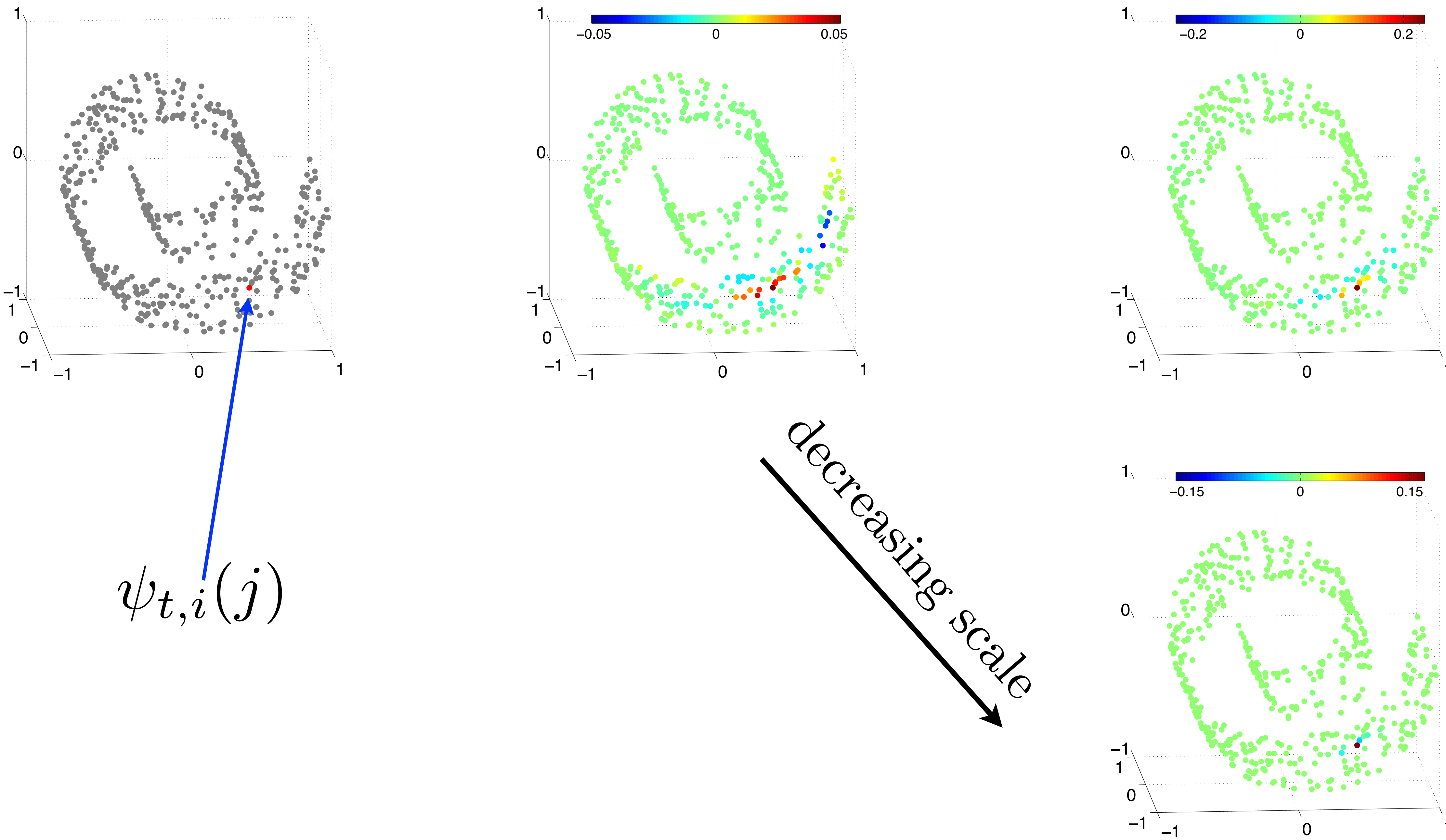
for any admissible kernel g

Scaling & Localization

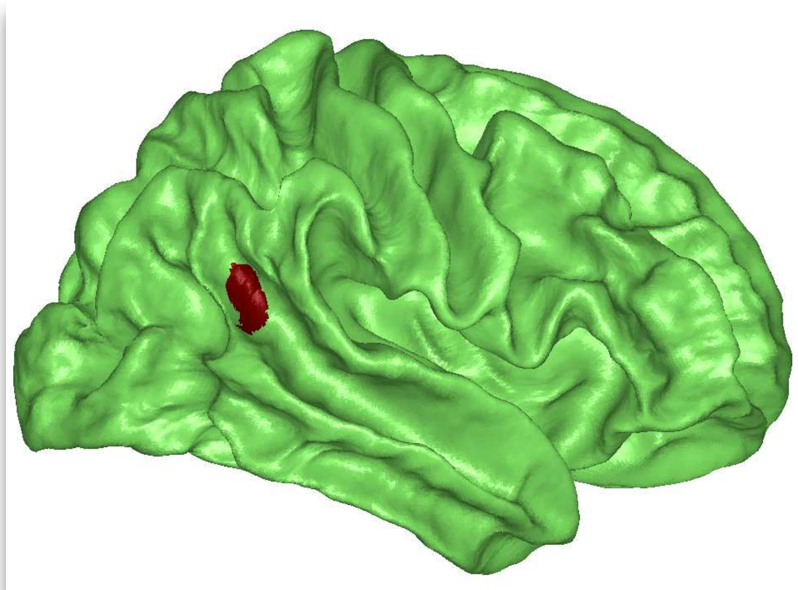


$\psi_{t,i}(j)$

Scaling & Localization



Example



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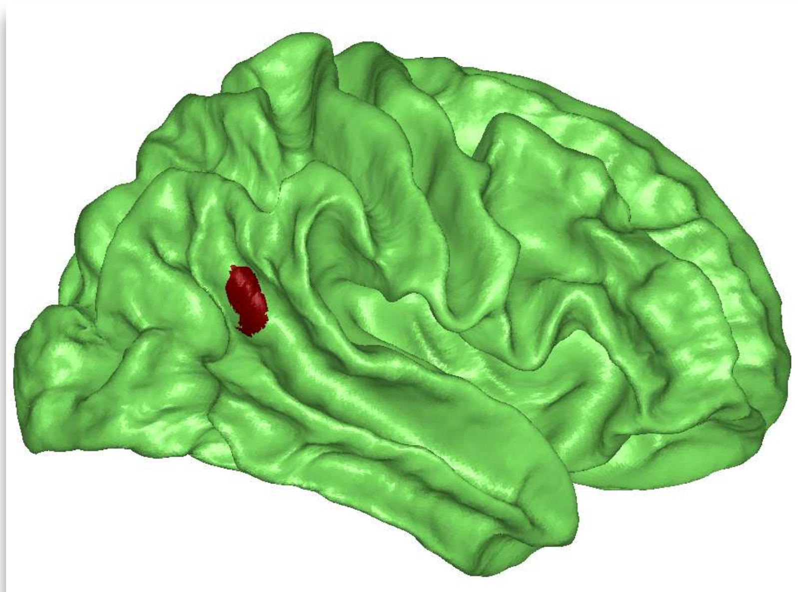

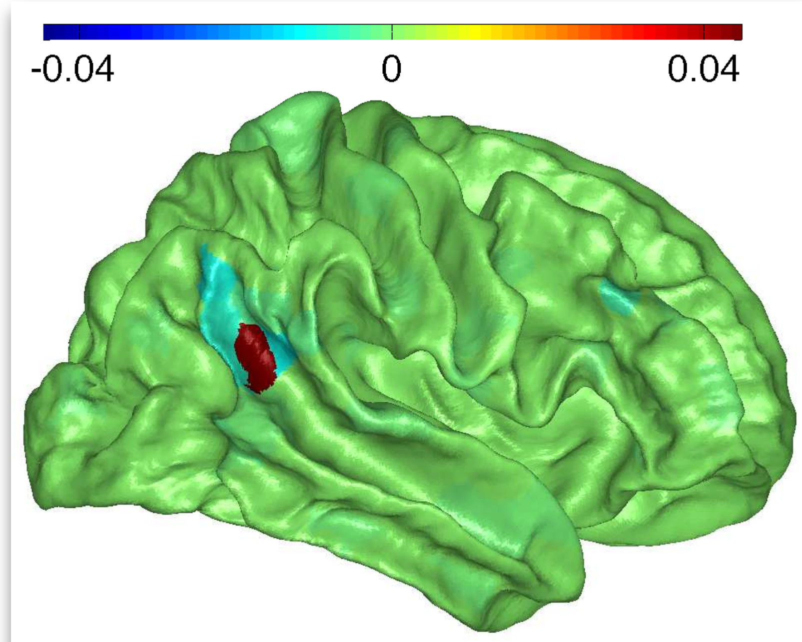

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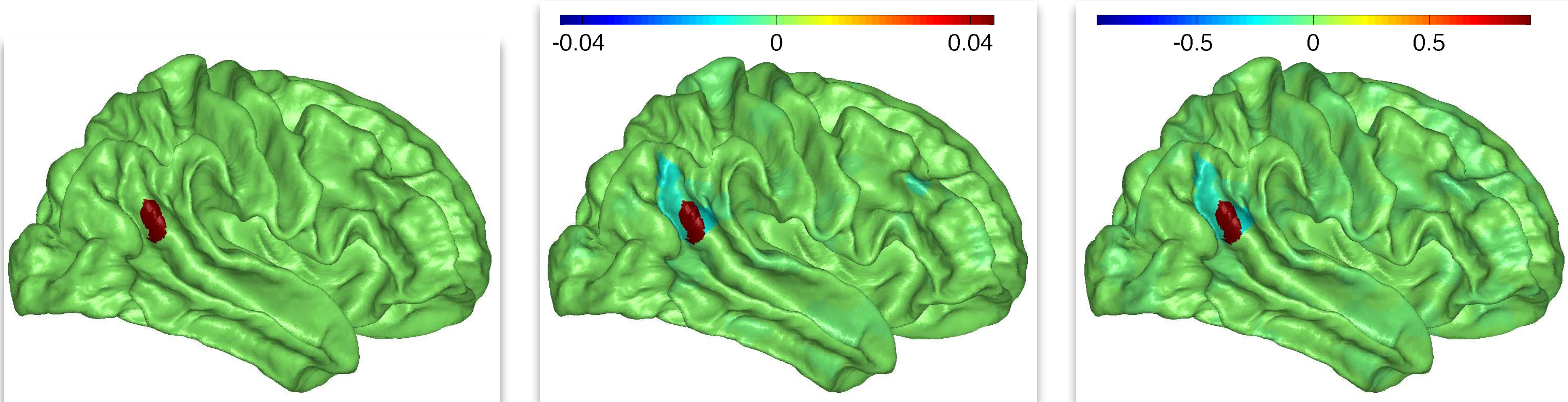

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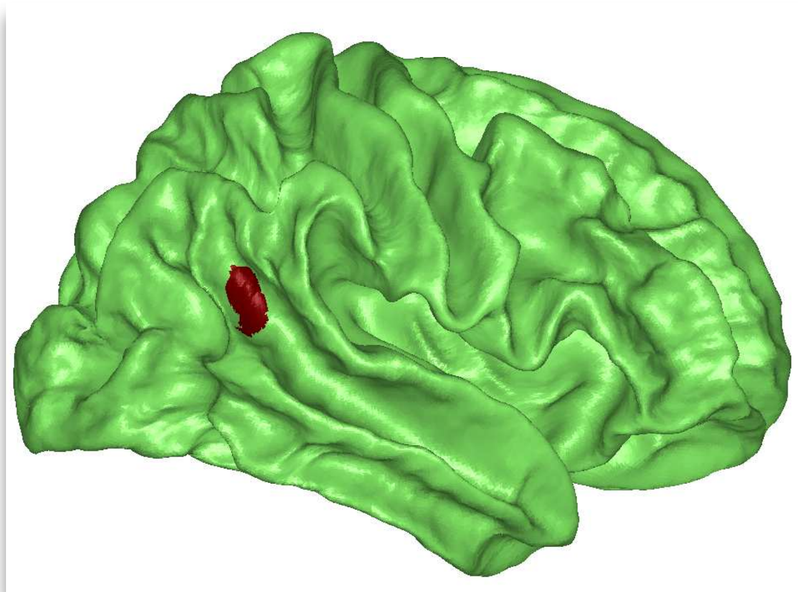

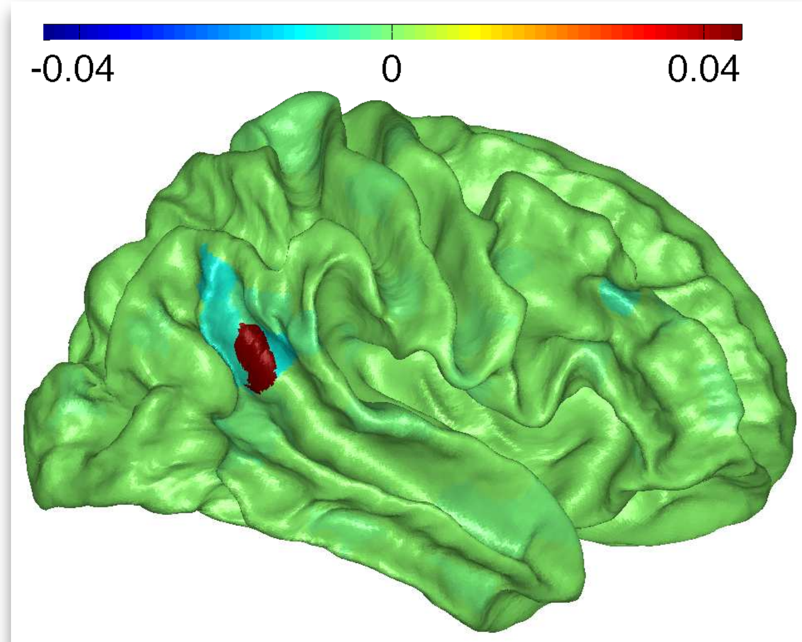

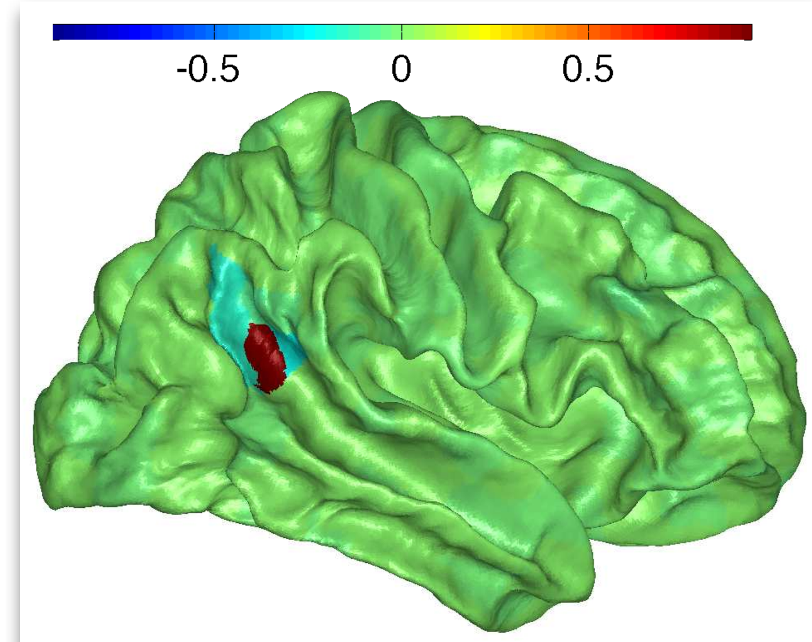

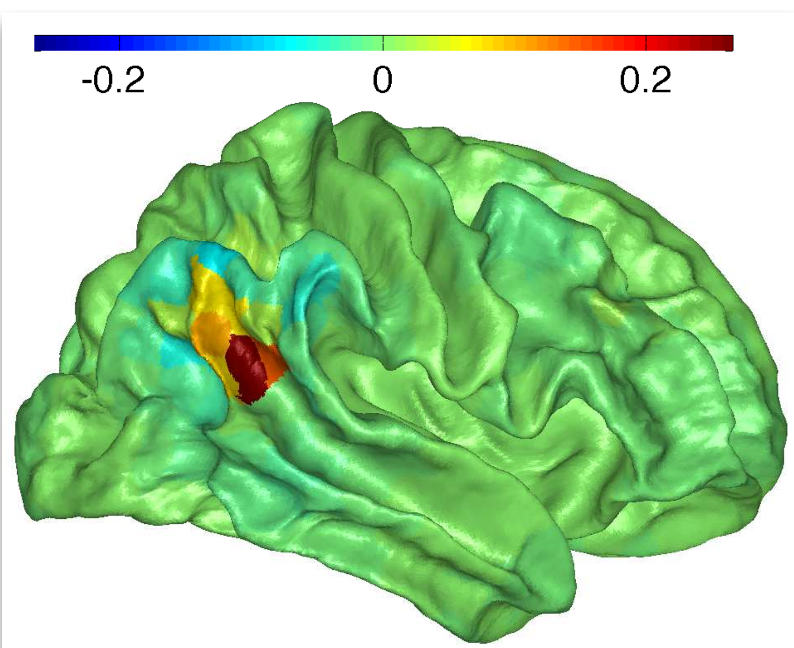

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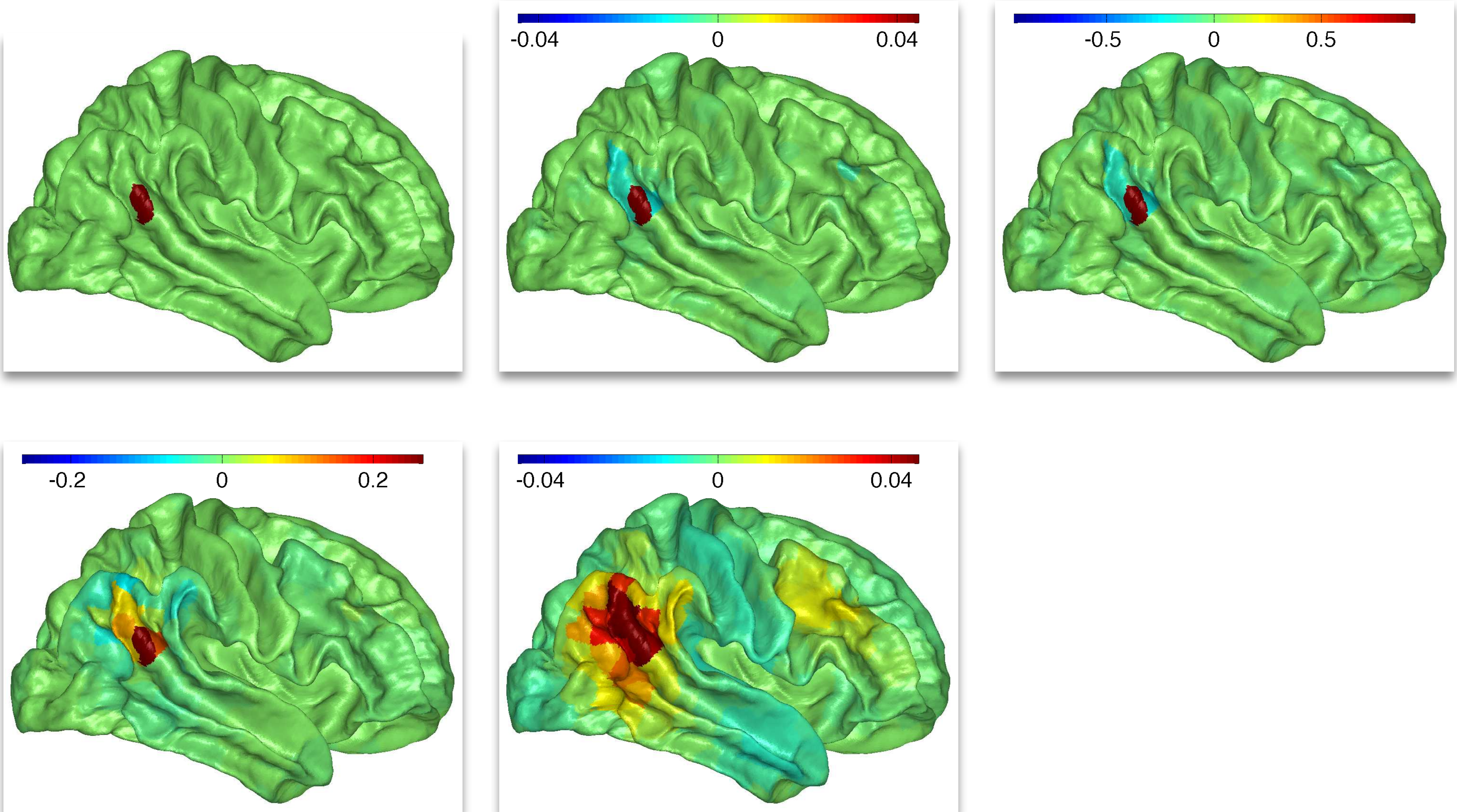

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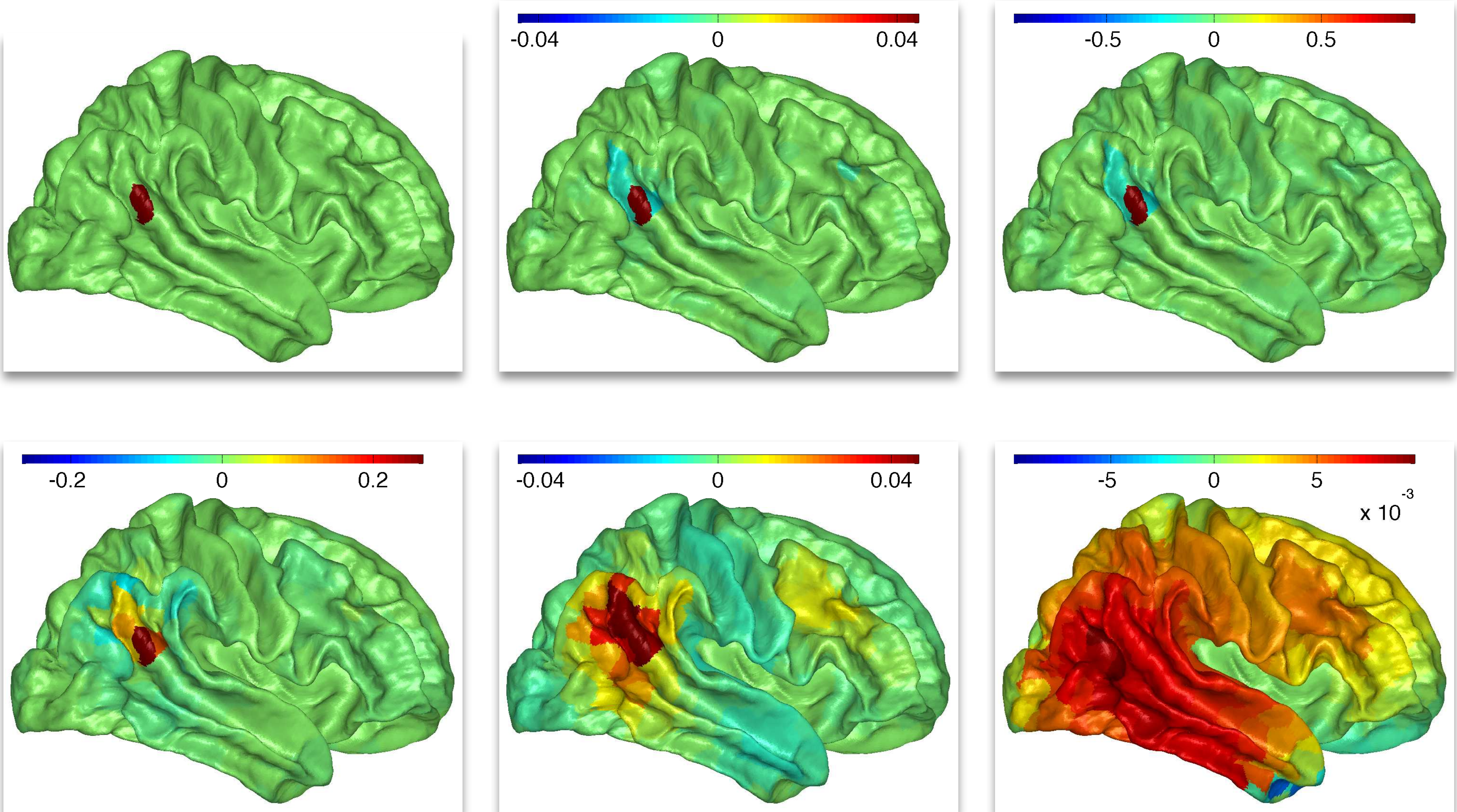
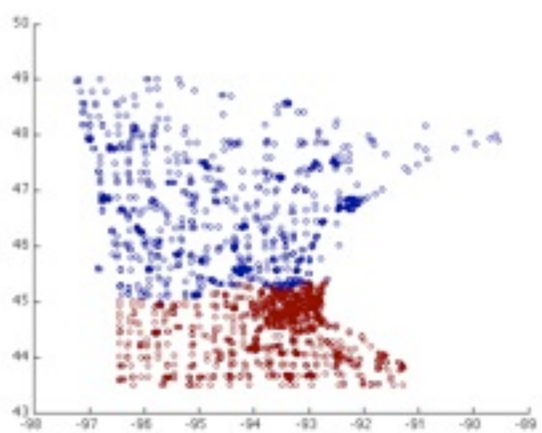
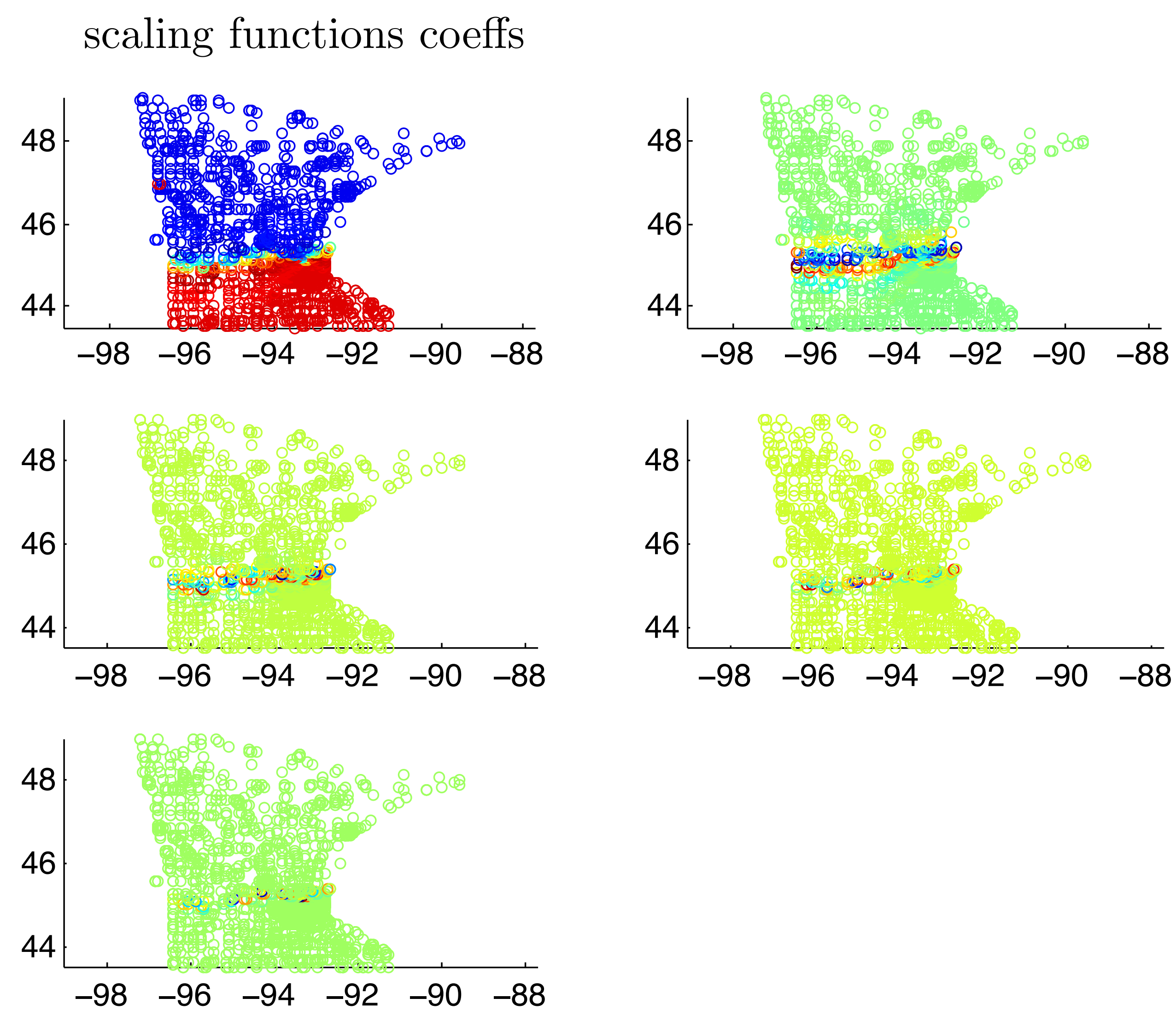


Sparsity and Smoothness on Graphs



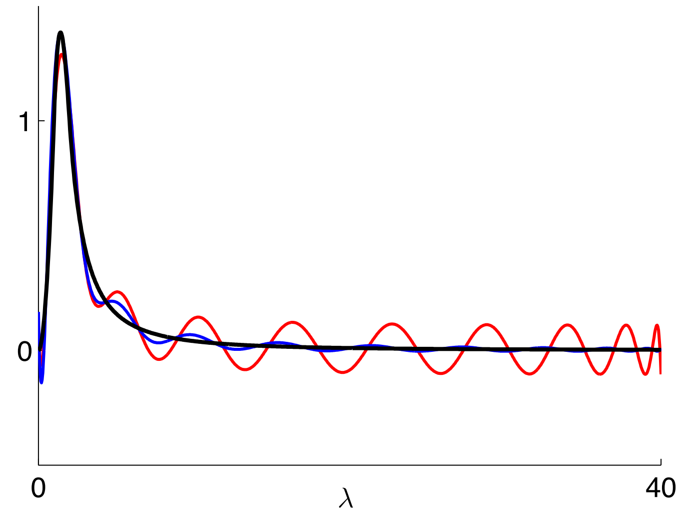


Remark on Implementation

Not necessary to compute spectral decomposition for filtering

Polynomial approximation : $$g(t\omega) \simeq \sum_{k=0}^{K-1} a_k(t) p_k(\omega)$$

ex: Chebyshev, minimax



It is to implement any Fourier multiplier

Then wavelet operator expressed with … acian:

$$T_g^t \simeq \sum_{k=0}^{K-1} a_k(t) \mathcal{L}^k$$

And use sparsity of Laplacian in an iterative way

Remark on Implementation

$$\tilde{W}_f(t,j) = \left(p(\mathcal{L})f^\#\right)_j \qquad |W_f(t,j) - \tilde{W}_f(t,j)| \leq B\|f\|$$

sup norm control (minimax or Chebyshef)

$$\tilde{W}_f(t_n,j) = \left(\frac{1}{2}c_{n,0}f^\# + \sum_{k=1}^{M_n} c_{n,k}\overline{T}_k(\mathcal{L})f^\#\right)_j$$

$$\overline{T}_k(\mathcal{L})f = \frac{2}{a_1}(\mathcal{L} - a_2 I)\left(\overline{T}_{k-1}(\mathcal{L})f\right) - \overline{T}_{k-2}(\mathcal{L})f$$

Computational cost dominated by matrix-vector multiply with (sparse) Laplacian matrix.

In particular $O(\sum_{n=1} M_n|E|)$

http://wiki.epfl.ch/sgwt

Note: "same" algorithm for adjoint !

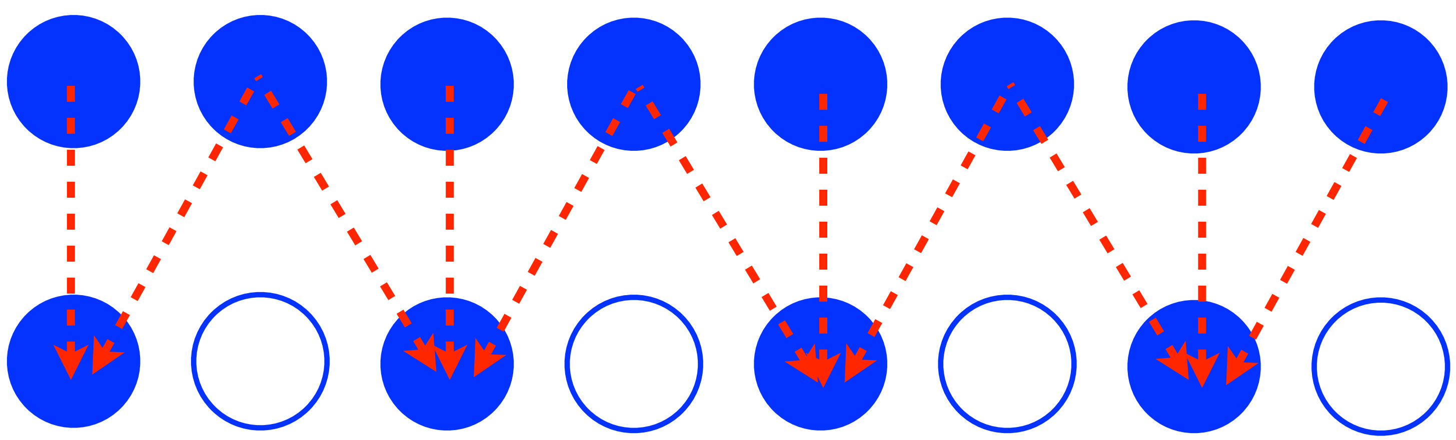
Graph wavelets

- Redundancy breaks sparsity
  - can we remove some or all of it ?
- Faster algorithms
  - traditional wavelets have fast filter banks implementation
  - whatever scale, you use the same filters
  - here: large scales -> more computations
- Goal: solve both problems at one

Basic Ingredients

Euclidean multiresolution is based on two main operations

- Filtering (typically low-pass and high-pass)
- Down and Up sampling



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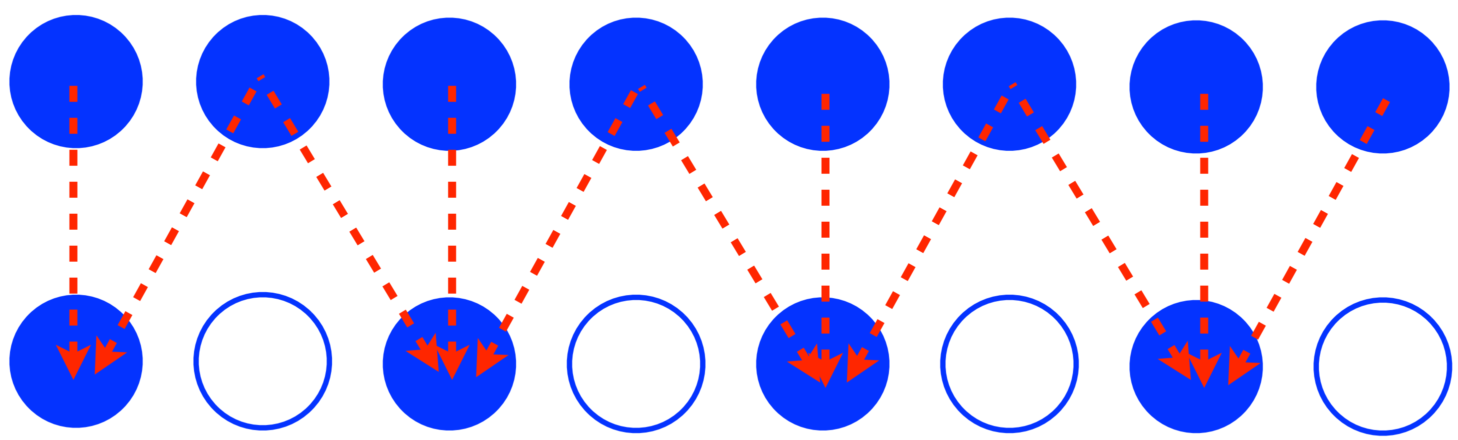

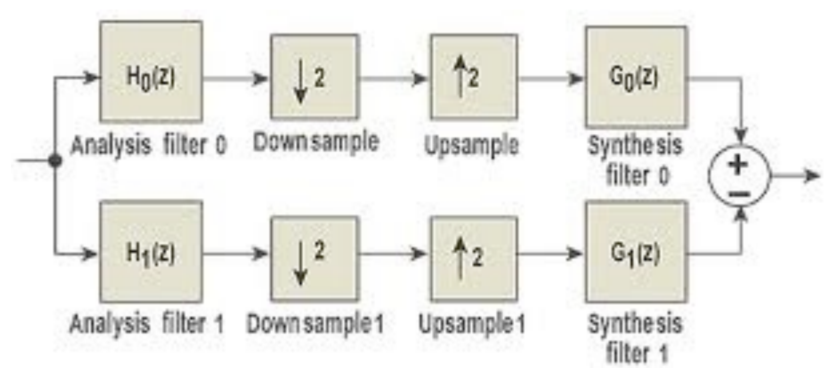

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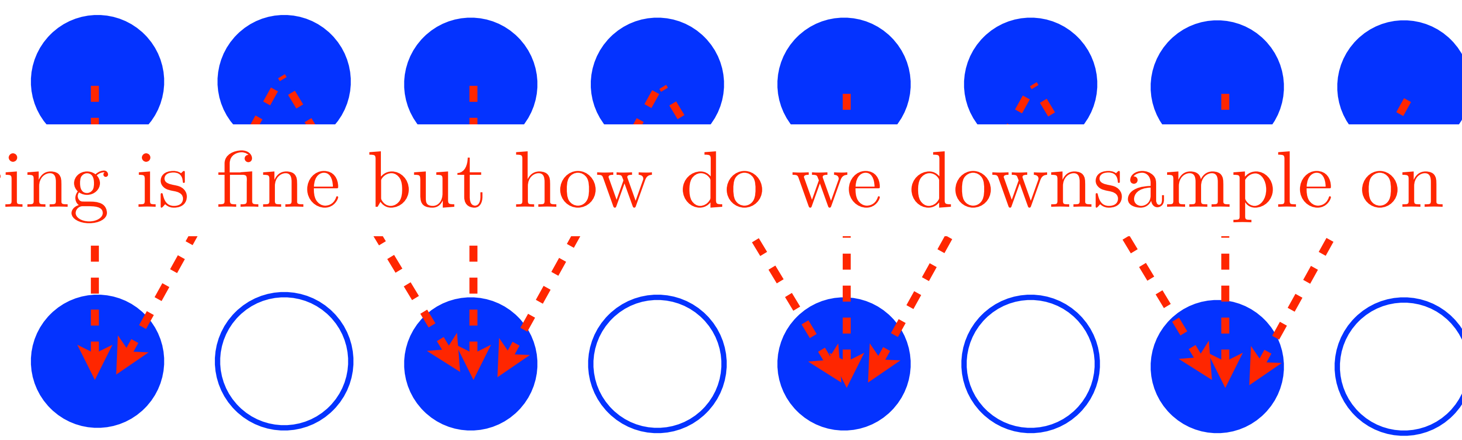
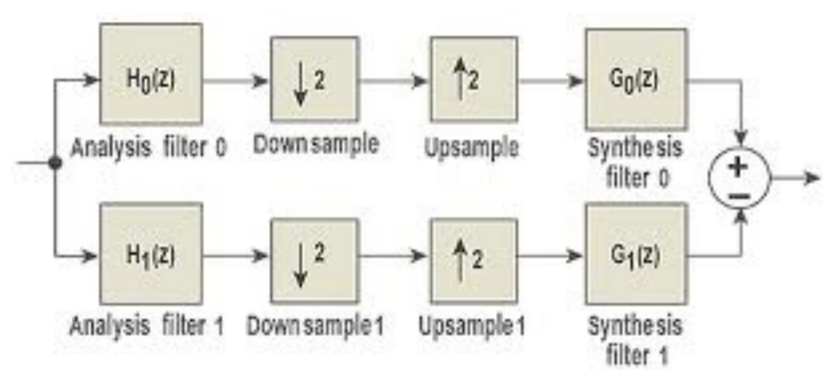
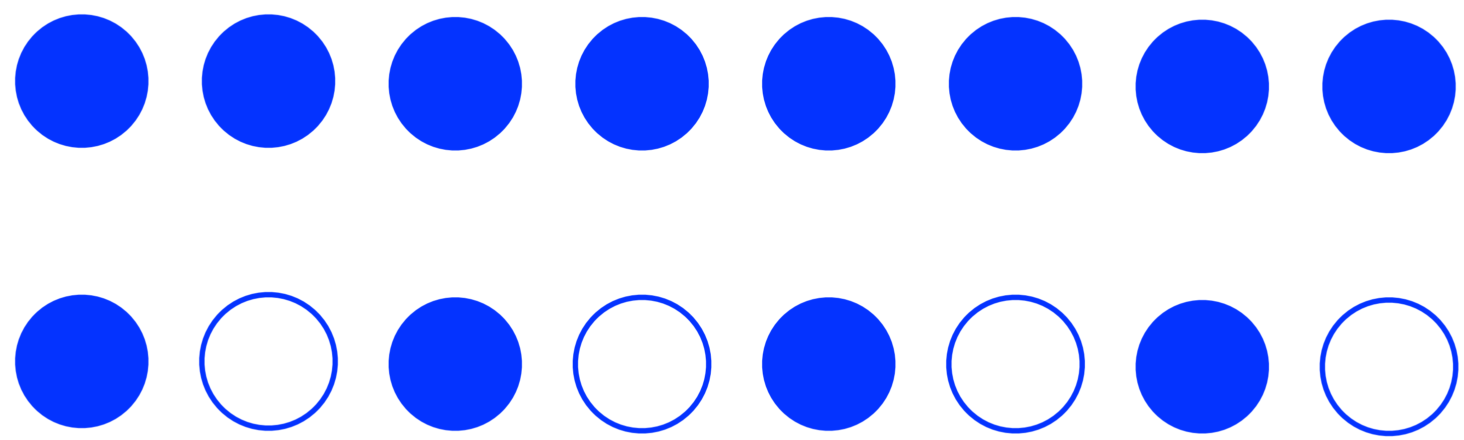


Filtering is fine but how do we downsample on graphs ???



Basic Ingredients

Subsampling is equivallent to splitting in two cosets (even, odd)



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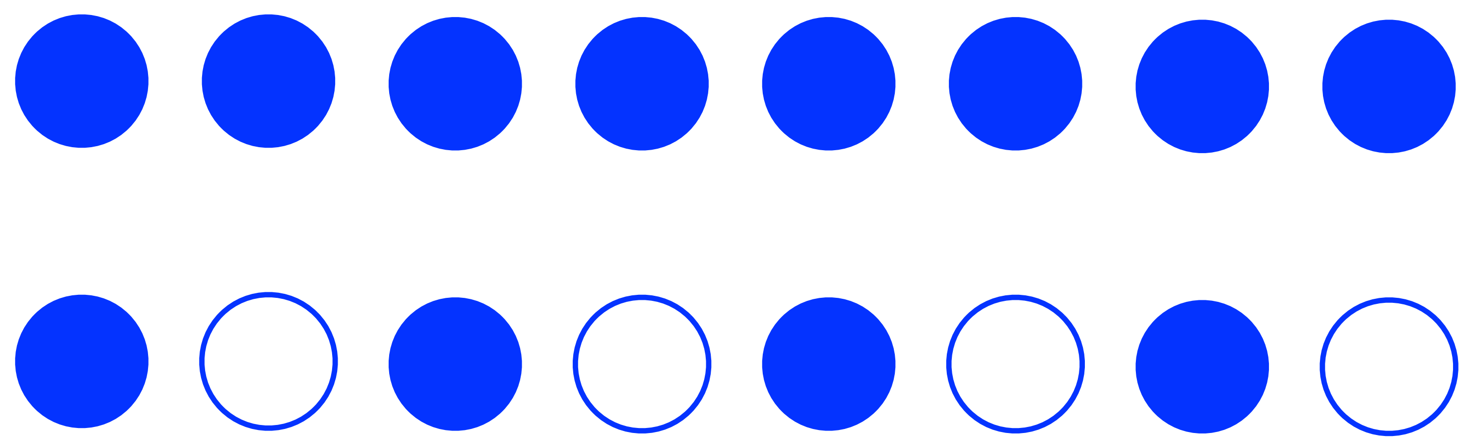


Questions: How do we partition a graph into meaningful cosets ?

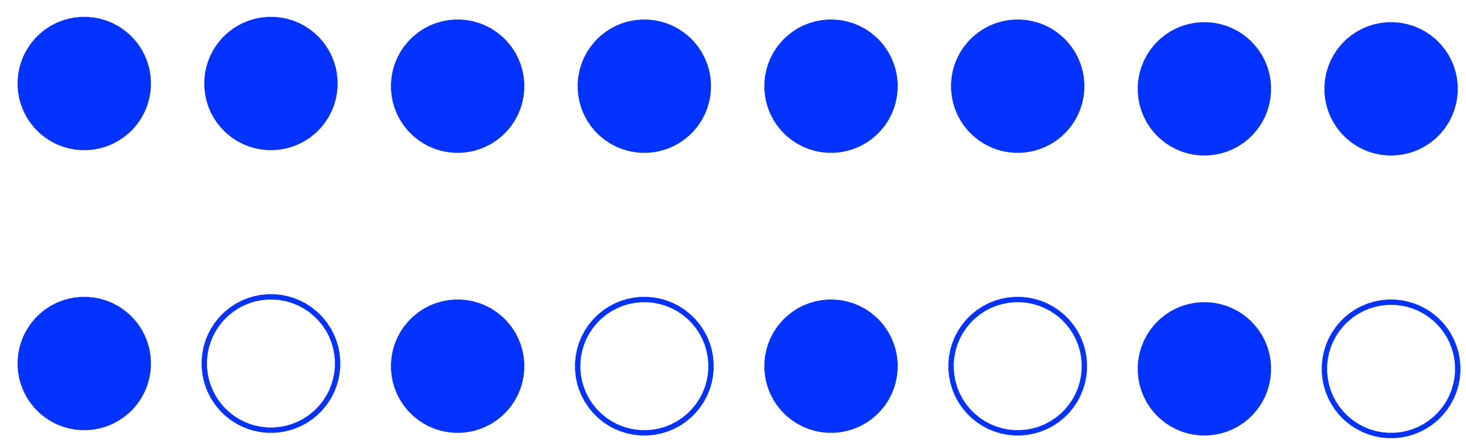
Are there efficient algorithms for these partitions ?

Are there theoretical guarantees ?

How do we define a new graph from the cosets ?

Cosets - A spectral view

Subsampling is equivallent to splitting in two cosets (even, odd)



Classically, selecting a coset can be interpreted easily in Fourier:

$$f_{\text{sub}}(i) = \frac{1}{2} f(i) \big(1 + \cos(\pi i)\big)$$

eigenvector of largest eigenvalue

Cosets and Nodal Domains

Nodal domain: maximally connected subgraph s.t. all vertices have same sign w.r.t a reference function

We would like to find a very large number of nodal domains, ideally $|V|$!

Nodal domains of Laplacian eigenvectors are special (and well studied)

Cosets and Nodal Domains

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We would like to find a very large number of nodal domains, ideally $|V|$!

Nodal domains of Laplacian eigenvectors are special (and well studied)

Theorem: the number of nodal domains associated to the largest laplacian eigenvector of a connected graph is maximal,

$$\nu(\phi_{\max}) = \nu(G) = |V|$$

IFF G is bipartite

In general: $\nu(G) = |V| - \chi(G) + 2$ (extreme cases: bipartite and complete graphs)

Cosets and Nodal Domains

Nodal domain: maximally connected subgraph s.t. all vertices have same sign w.r.t a reference function

We would like to find a very large number of nodal domains, ideally $|V|$!

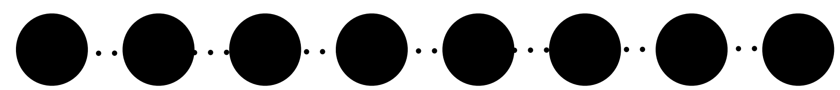
Nodal domains of Laplacian eigenvectors are special (and well studied)

For any connected graph we will thus naturally define cosets and their associated selection functions

$$V_+ = \{i \in V \text{ s.t. } \phi_{N-1}(i) \geq 0\} \qquad V_- = \{i \in V \text{ s.t. } \phi_{N-1}(i) < 0\}$$

$$M_+(i) = \frac{1}{2}\big(1 + \text{sgn}(\phi_{N-1}(i))\big) \qquad M_-(i) = \frac{1}{2}\big(1 - \text{sgn}(\phi_{N-1}(i))\big)$$

Examples of cosets

Simple line graph 

$\phi_k(u) = \sin(\pi k u/n + \pi/2n) \qquad \lambda_k = 2 - 2\cos(\pi k/n) \qquad 1 \leq k \leq n$

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Simple line graph

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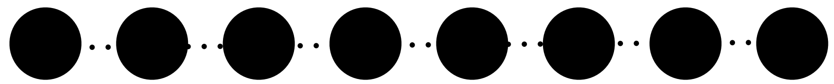
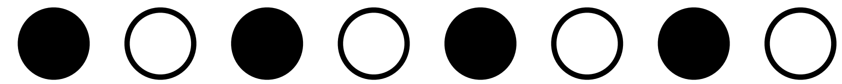
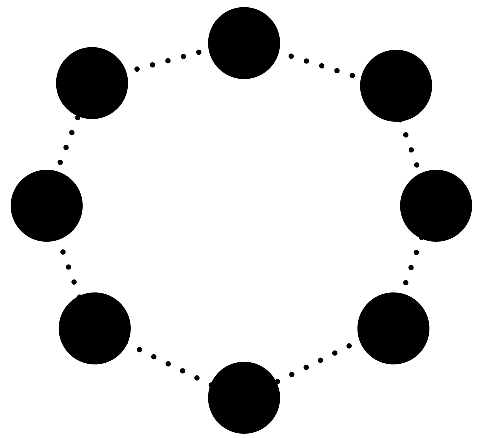




Simple ring graph



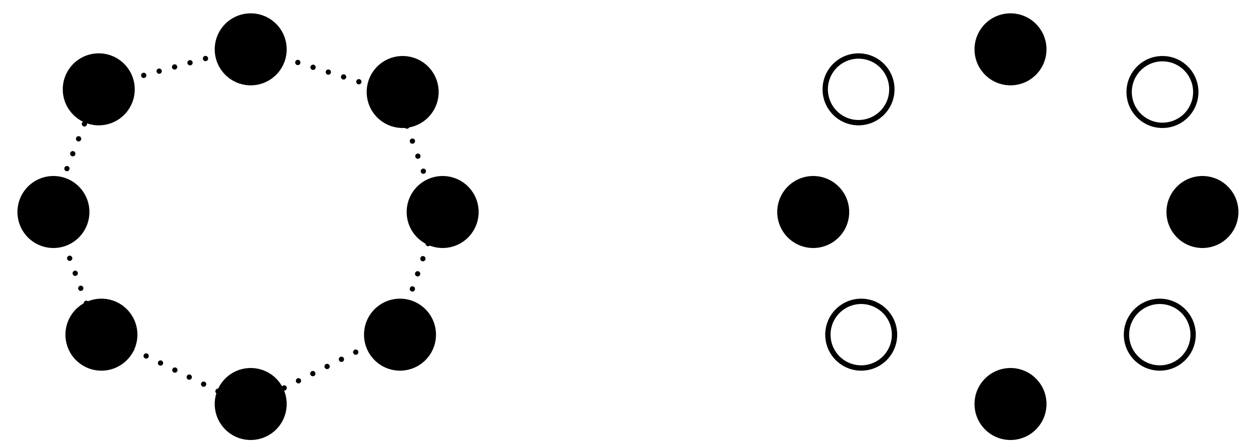
$$\phi_k^1(u) = \sin(2\pi k u / n) \qquad \phi_k^2(u) = \cos(2\pi k u / n) \quad 1 \leq k \leq n/2$$

$$\lambda_k = 2 - 2\cos(2\pi k / n)$$

Examples of cosets

Simple line graph

Simple ring graph



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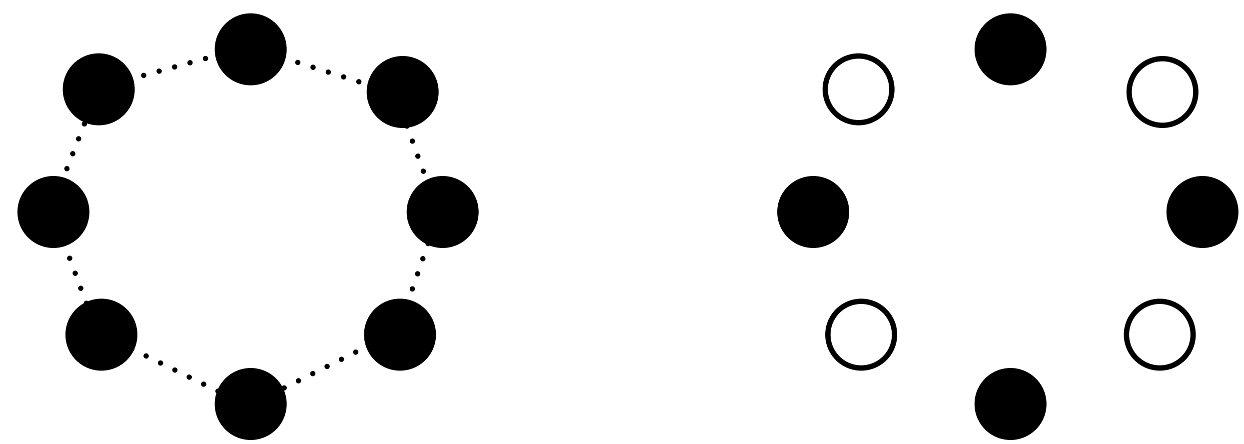
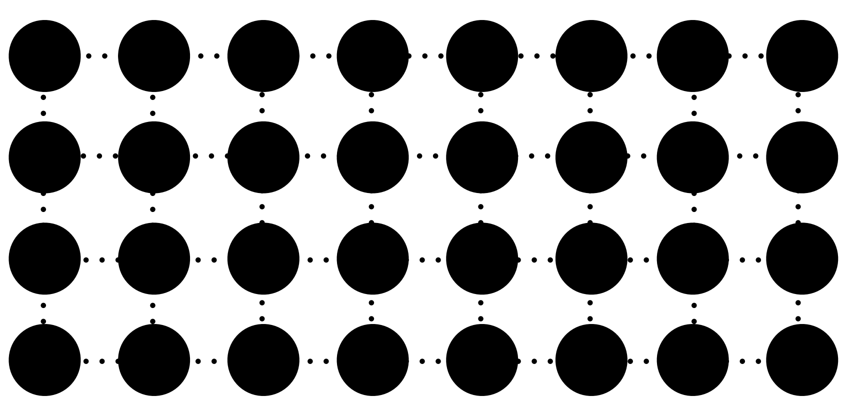


Lattice



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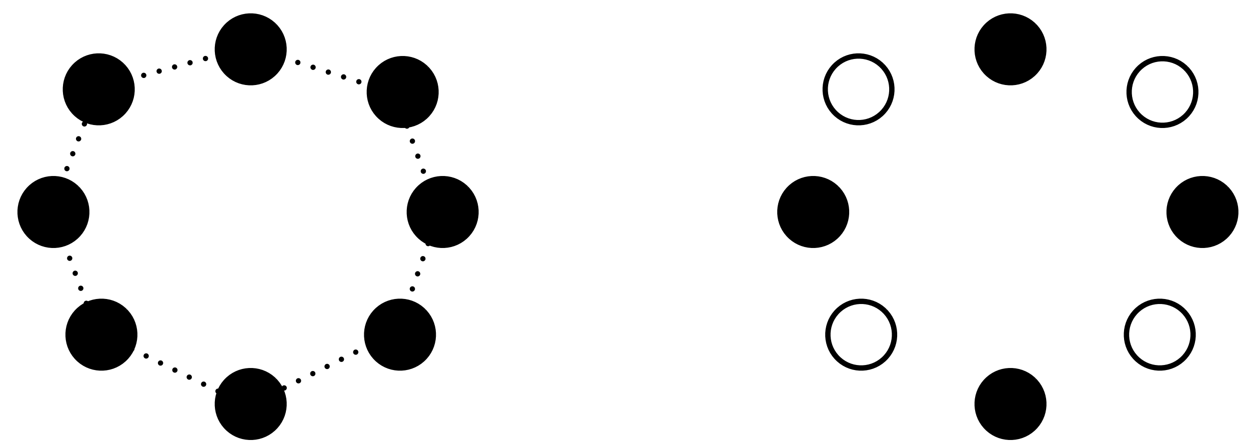

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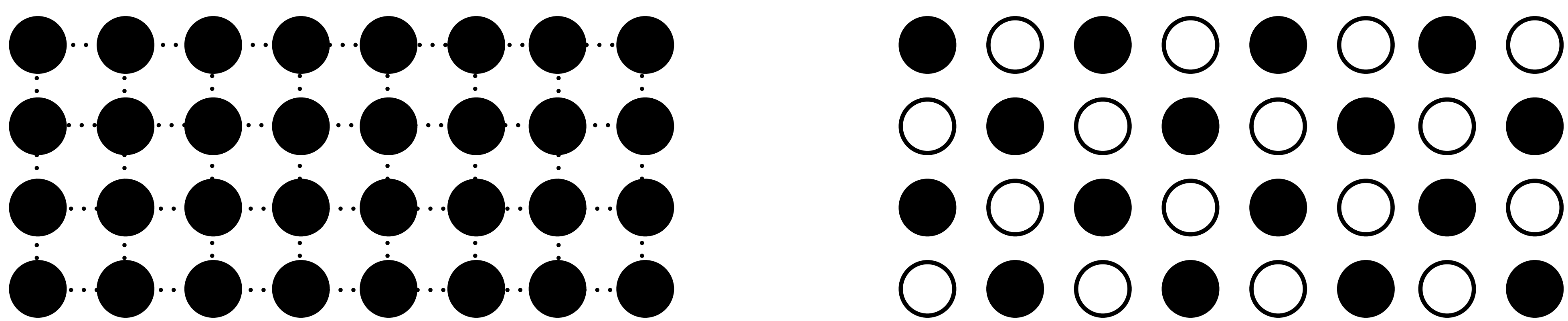


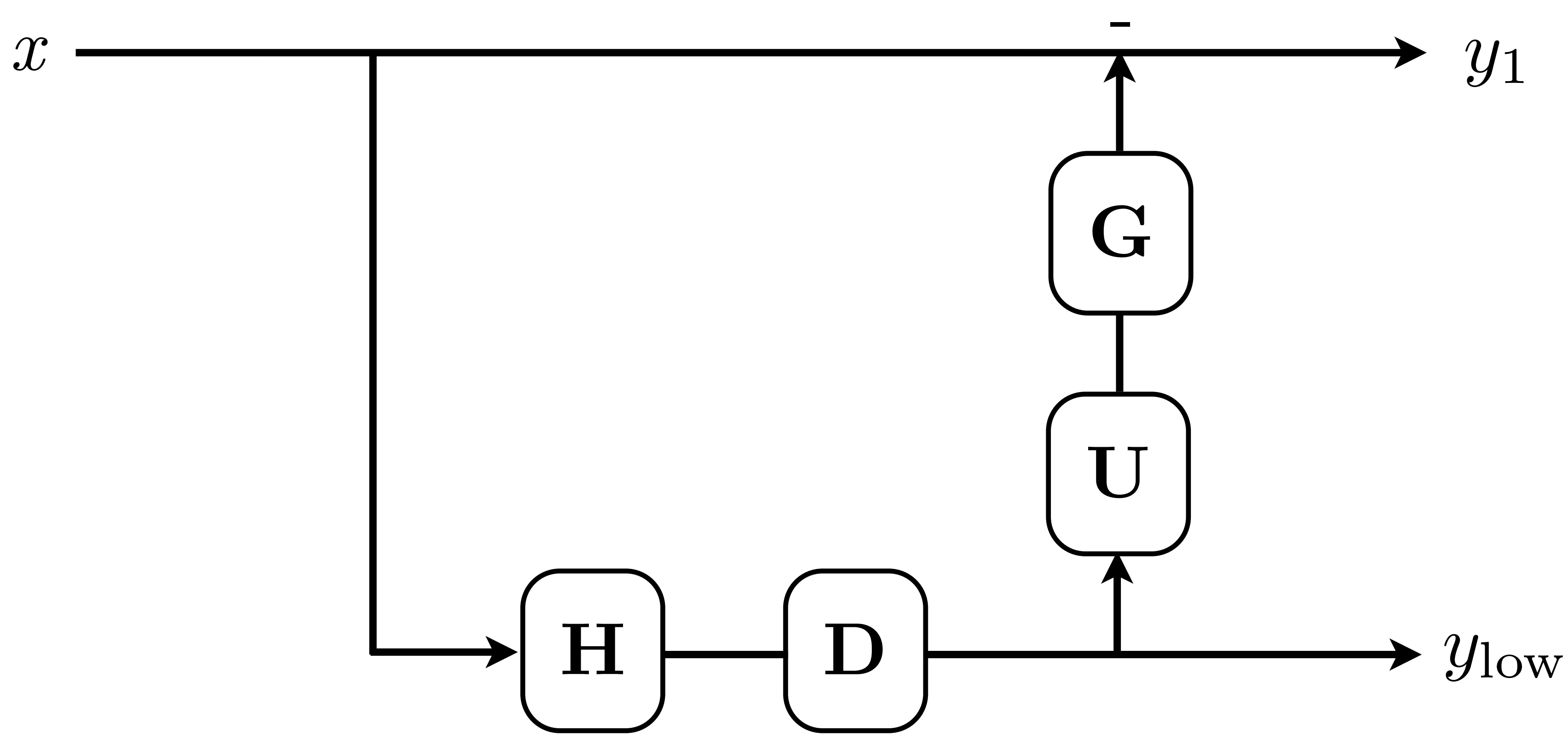
quincunx

The Agonizing Limits of Intuition

- Multiplicity of $\lambda_{\max}$
  - how do we choose the control vector in that subspace ?
  - even a prescription can be numerically ill-defined
  - graphs with "flat" spectrum in close to their spectral radius
- Laplacian eigenvectors do not always behave like global oscillations
  - seems to be true for random perturbations of simple graphs
  - true even for a class of trees [Saito2011]

The Laplacian Pyramid

Analysis operator



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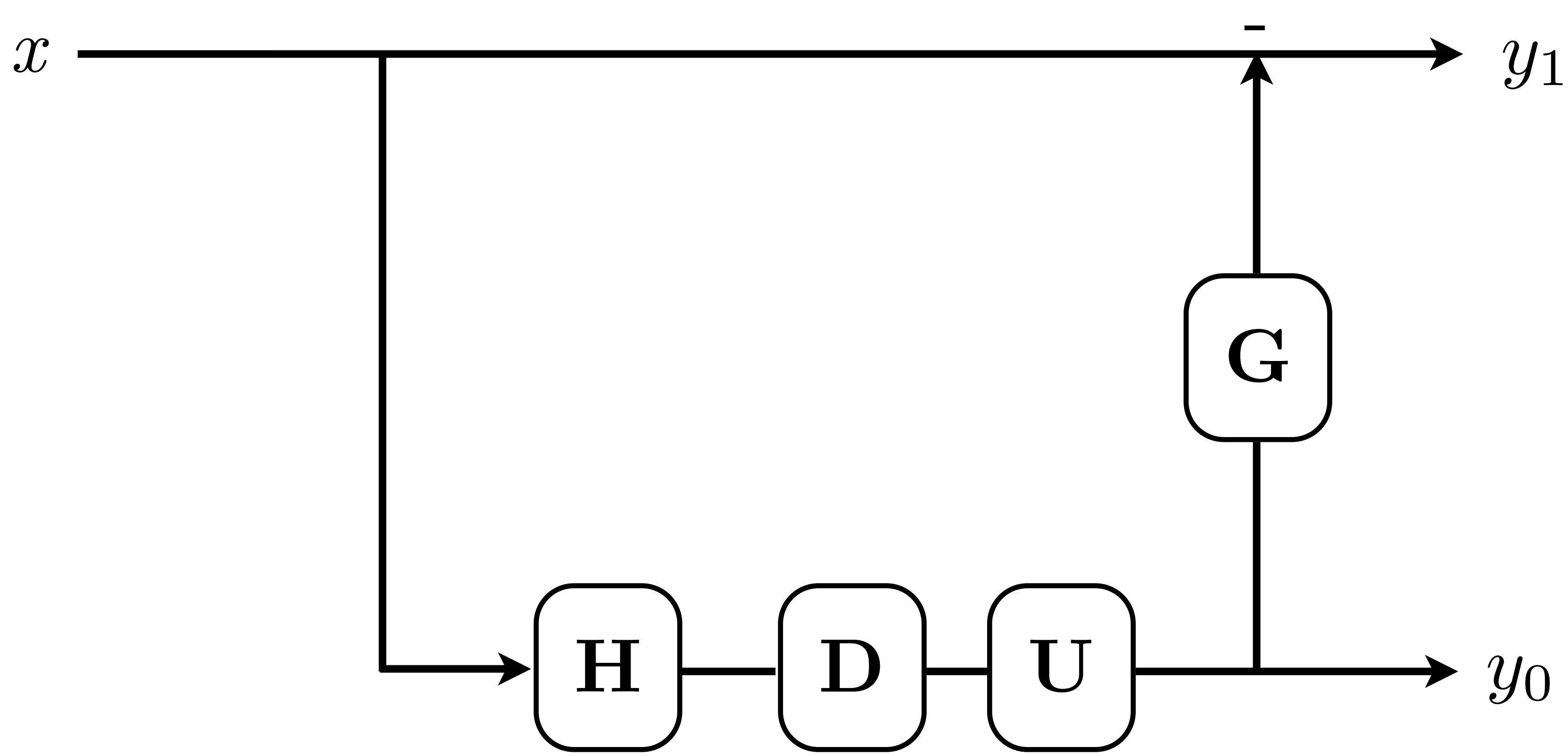

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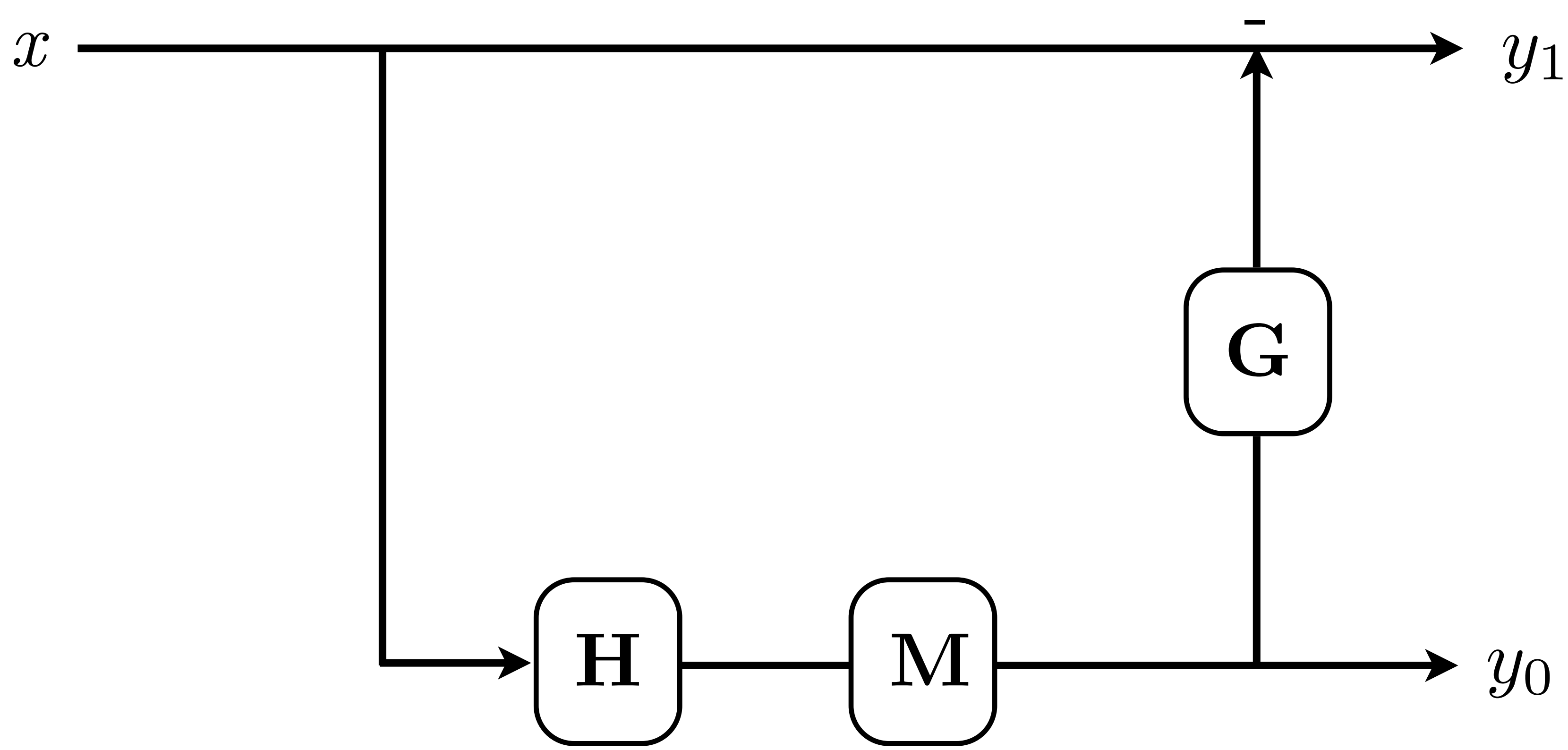

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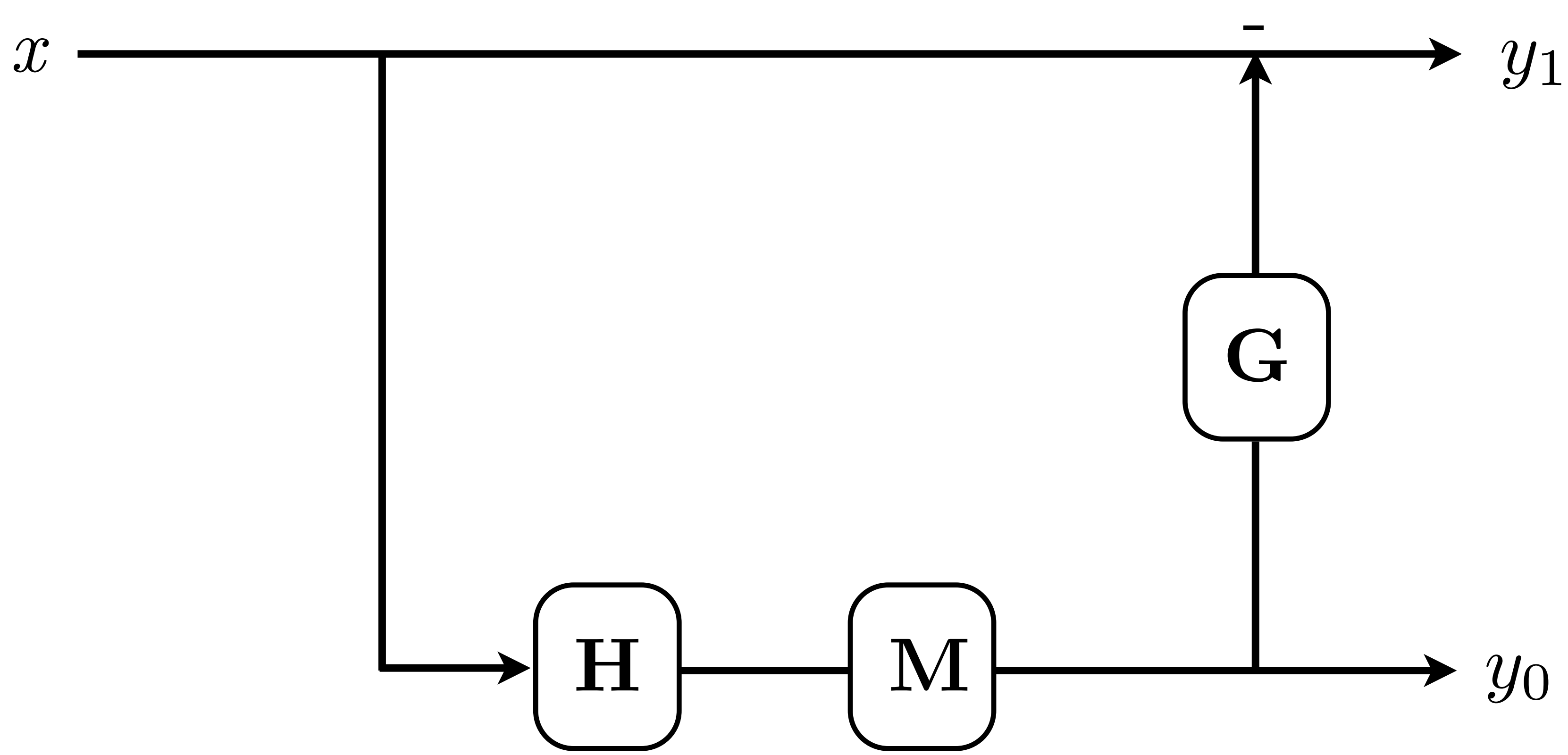

$$
\begin{aligned}
y_0 &= \mathbf{H_m} x \\
&= \mathbf{MH} x
\end{aligned}
\qquad\qquad
\begin{aligned}
y_1 &= x - \mathbf{G} y_0 \\
&= x - \mathbf{G H_m} x
\end{aligned}
$$

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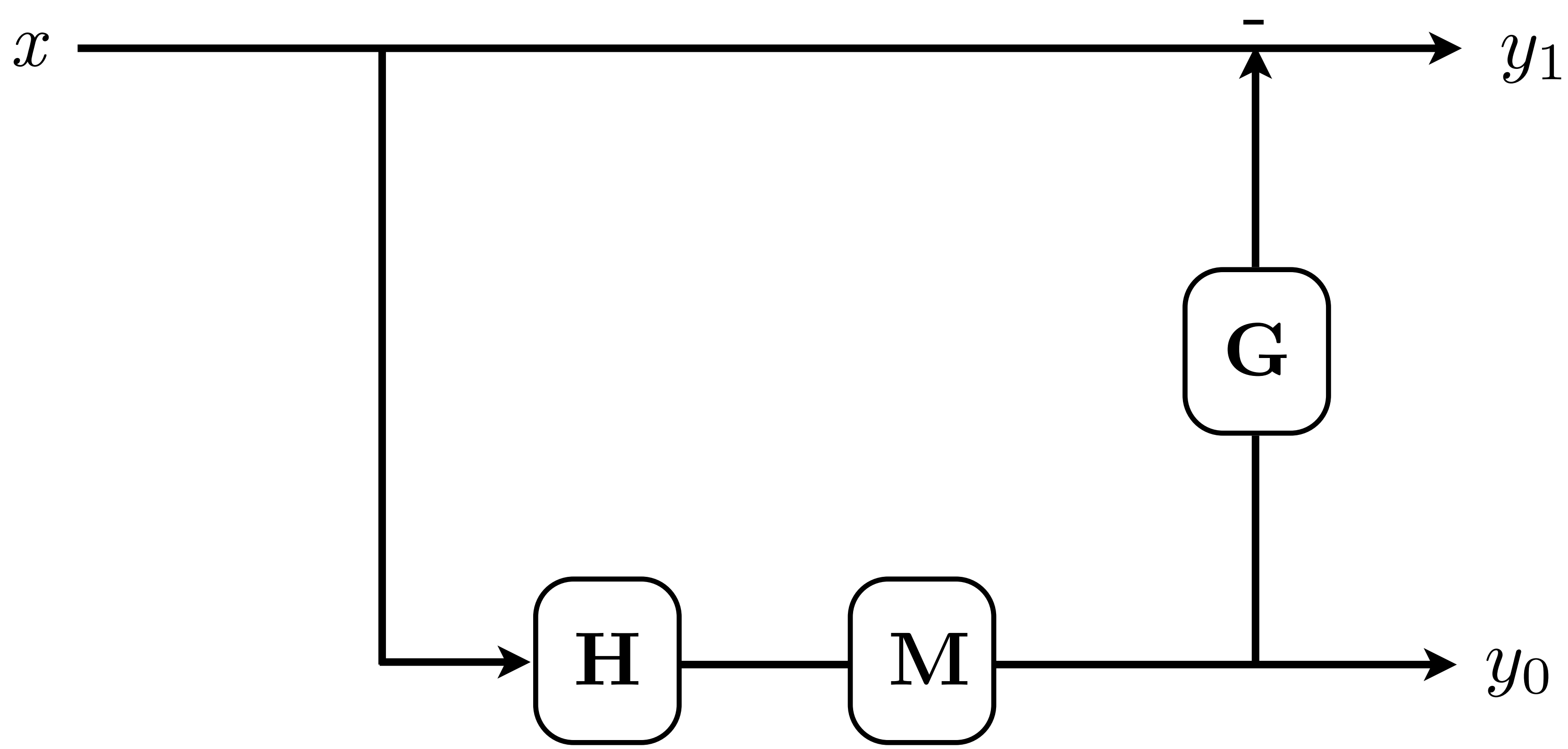

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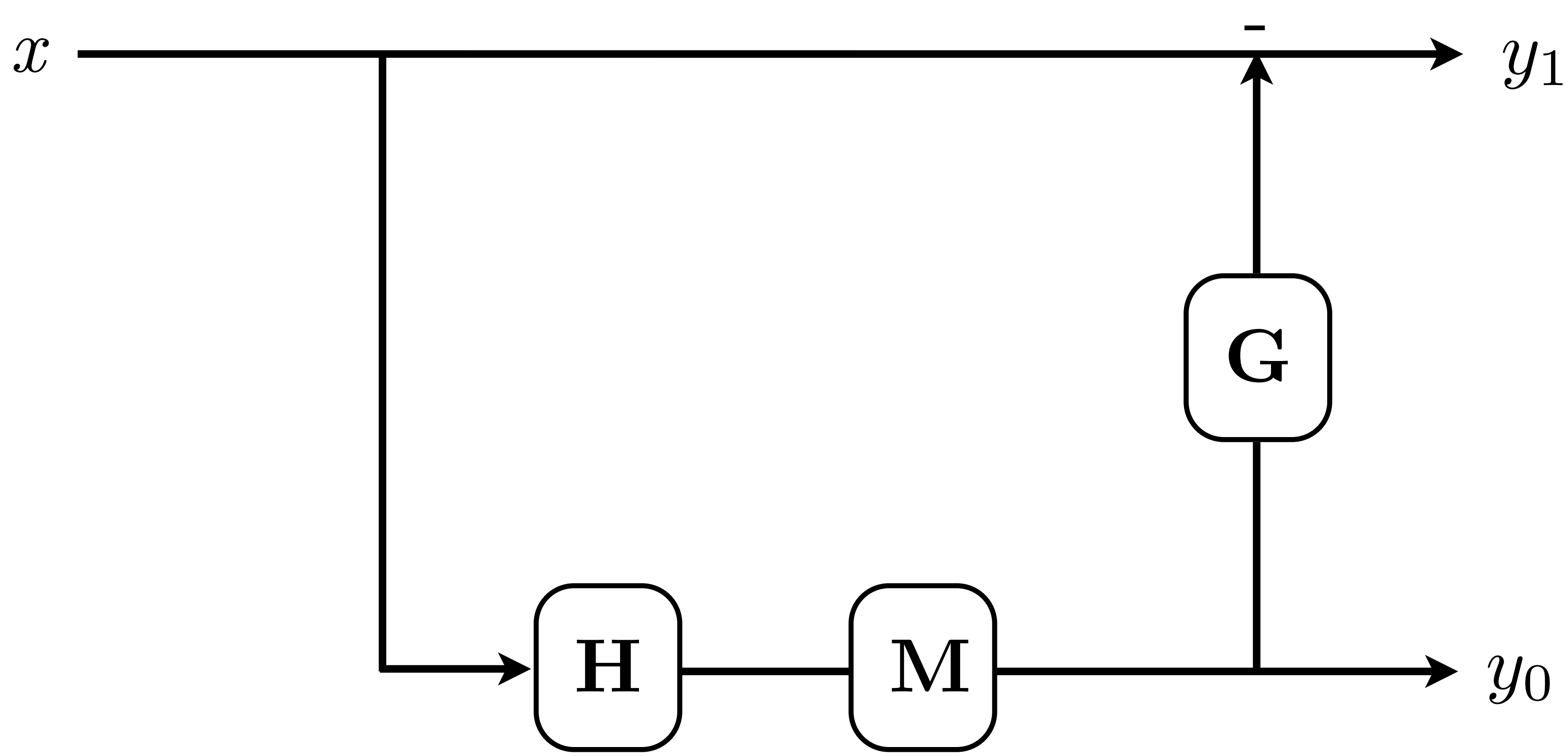

$$\underbrace{\begin{pmatrix} y_0 \\ y_1 \end{pmatrix}}_{y} = \underbrace{\begin{pmatrix} \mathbf{H_m} \\ \mathbf{I} - \mathbf{G}\mathbf{H_m} \end{pmatrix}}_{\mathbf{T_a}} x,$$

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Simple (traditional) left inverse

$$\hat{x} = \underbrace{\begin{pmatrix} \mathbf{G} & \mathbf{I} \end{pmatrix}}_{\mathbf{T_s}} \underbrace{\begin{pmatrix} y_0 \\ y_1 \end{pmatrix}}_{y}$$

$\mathbf{T_s}\mathbf{T_a} = \mathbf{I}$ with no conditions on $\mathbf{H}$ or $\mathbf{G}$

The Laplacian Pyramid

Pseudo Inverse ?

$$\mathbf{T_a}^{\dagger} = \left(\mathbf{T_a}^T \mathbf{T_a}\right)^{-1} \mathbf{T_a}^T$$

Let's try to use only filters

The Laplacian Pyramid

Pseudo Inverse ?

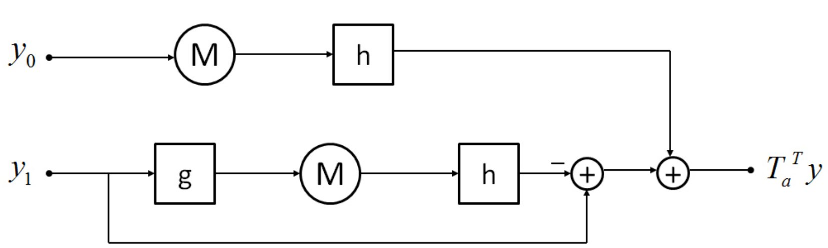
$$\mathbf{T_a}^{\dagger} = \left(\mathbf{T_a}^T \mathbf{T_a}\right)^{-1} \mathbf{T_a}^T$$

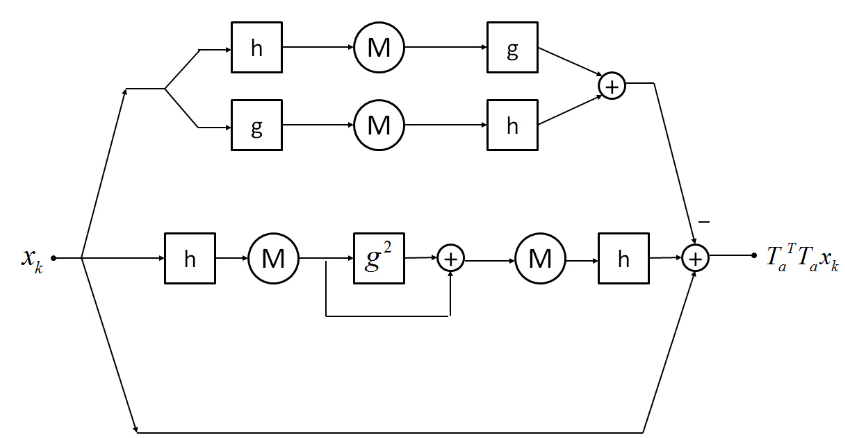
Let's try to use only filters

Define iteratively, through descent on LS:

$$\arg\min_x \|\mathbf{T_a} x - y\|_2^2 \longrightarrow \hat{x}_{k+1} = \hat{x}_k + \tau \mathbf{T_a}^T (y - \mathbf{T_a} \hat{x}_k)$$

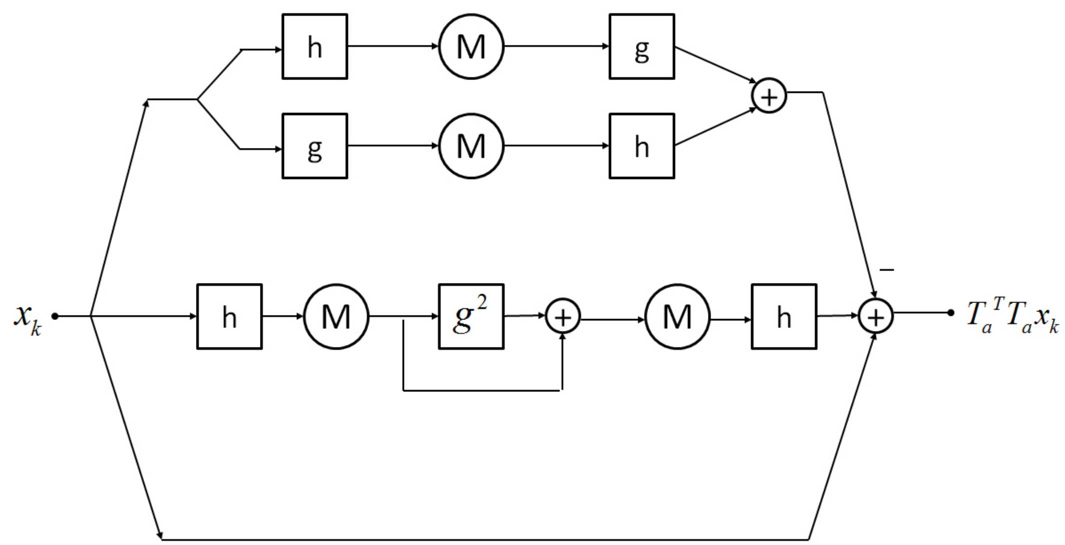
$$\mathbf{T_a}^T = (\mathbf{H_m}^T \quad \mathbf{I} - \mathbf{H_m}^T \mathbf{G}^T)$$





The Laplacian Pyramid

we can easily implement $\mathbf{T_a}^T\mathbf{T_a}$ with filters and masks:



With the real symmetric matrix $\mathbf{Q} = \mathbf{T_a}^T\mathbf{T_a}$ and $b = \mathbf{T_a}^T y$

$$x_N = \tau \sum_{j=0}^{N-1} (\mathbf{I} - \tau\mathbf{Q})^j b$$

Use Chebyshev approximation of: $L(\omega) = \tau \sum_{j=0}^{N-1} (1 - \tau\omega)^j$

Kron Reduction

In order to iterate the construction, we need to construct a graph on the reduced vertex set.

$$\mathbf{A}_{\mathrm{r}} = \mathbf{A}[\alpha, \alpha] - \mathbf{A}[\alpha, \alpha)\mathbf{A}(\alpha, \alpha)^{-1}\mathbf{A}(\alpha, \alpha]$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}[\alpha, \alpha] & \mathbf{A}[\alpha, \alpha) \\ \mathbf{A}(\alpha, \alpha] & \mathbf{A}(\alpha, \alpha) \end{bmatrix}$$

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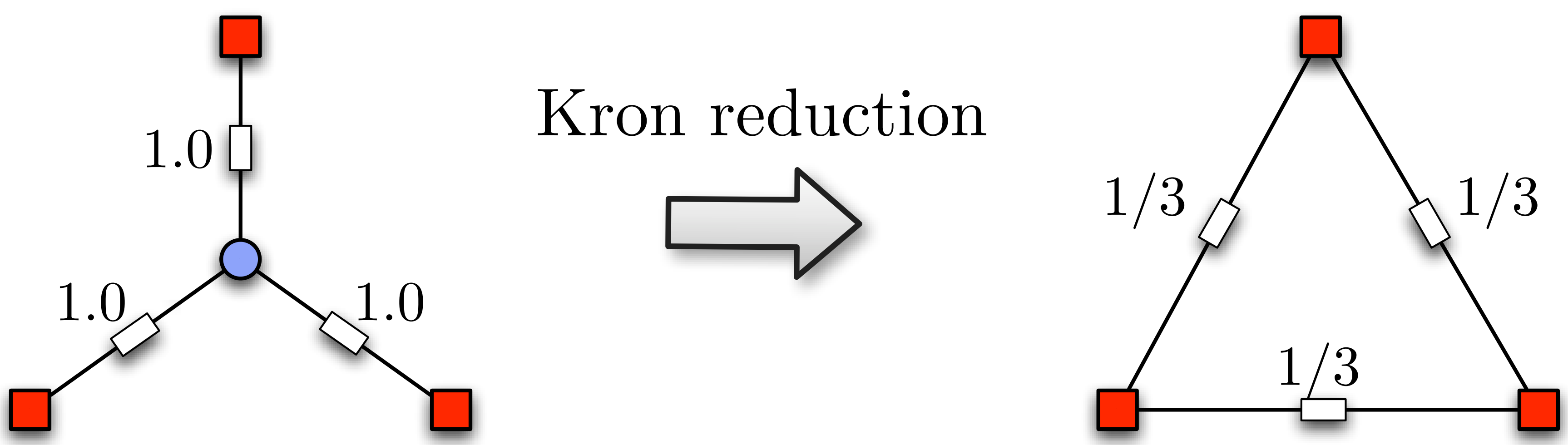

[Dorfler et al, 2011]

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Properties:

maps a weighted undirected laplacian to a weighted undirected laplacian

spectral interlacing (spectrum does not degenerate)

$$\lambda_k(\mathbf{A}) \leq \lambda_k(\mathbf{A}_r) \leq \lambda_{k+n-|\alpha|}(\mathbf{A})$$

disconnected vertices linked in reduced graph IFF there is a path that runs only through eliminated nodes

Example

Note: For a k-regular bipartite graph

$$\mathbf{L} = \begin{bmatrix} k\mathbf{I}_n & -\mathbf{A} \\ -\mathbf{A}^T & k\mathbf{I}_n \end{bmatrix}$$

Kron-reduced Laplacian: $\mathbf{L}_r = k^2\mathbf{I}_n - \mathbf{A}\mathbf{A}^T$

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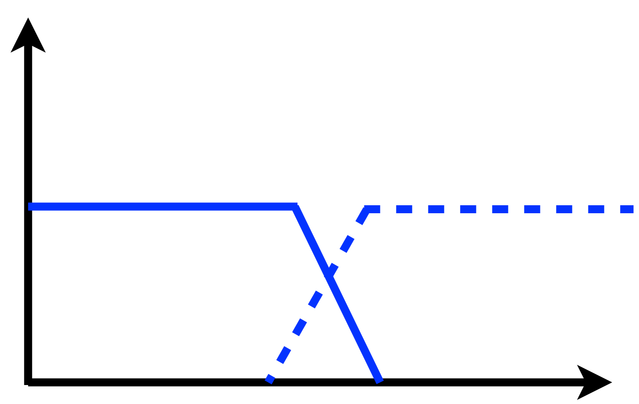
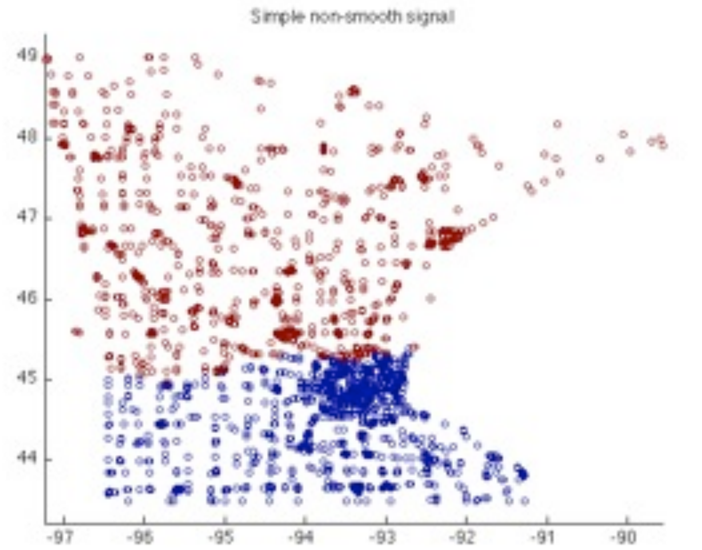
$$\hat{f}_r(i) = \hat{f}(i) + \hat{f}(N-i) \quad i = 1, ..., N/2$$




Simple non-smooth signal
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-97
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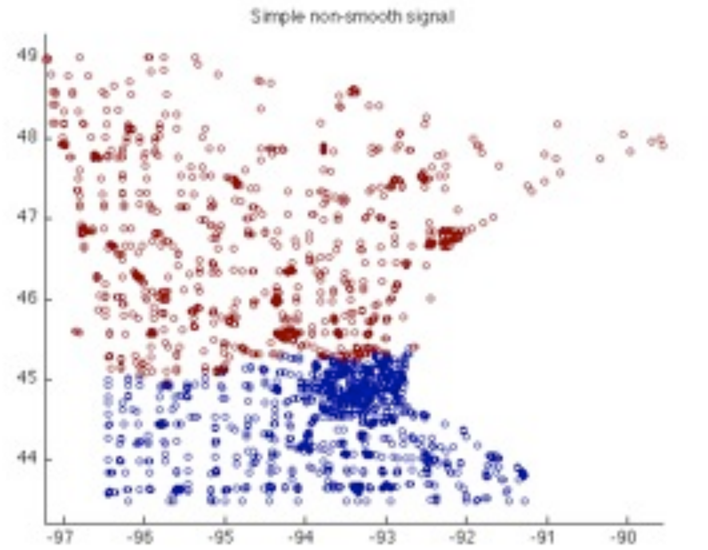
-93
-92
-91
-90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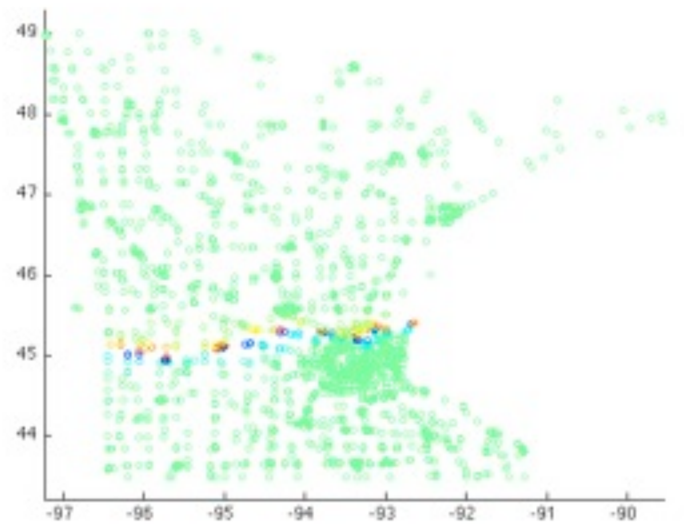

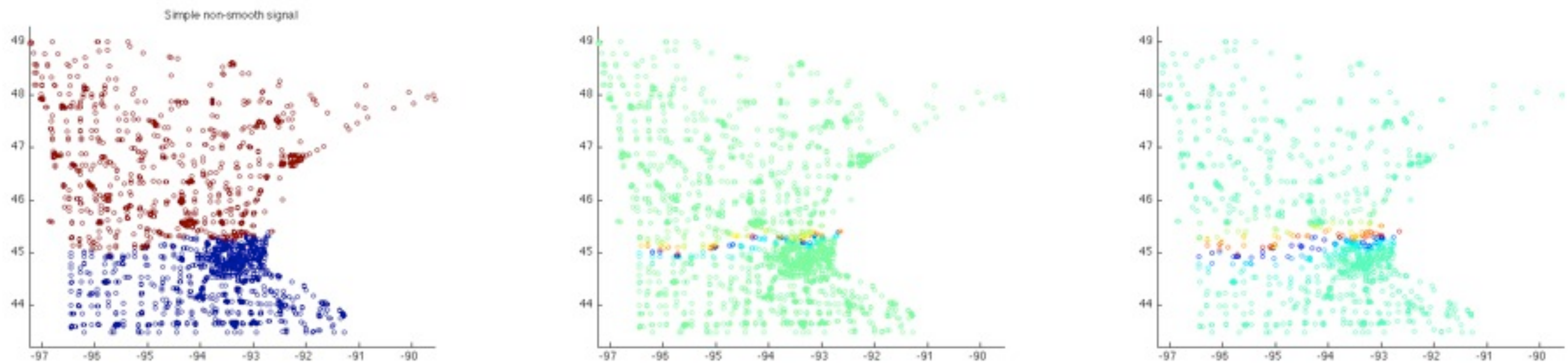
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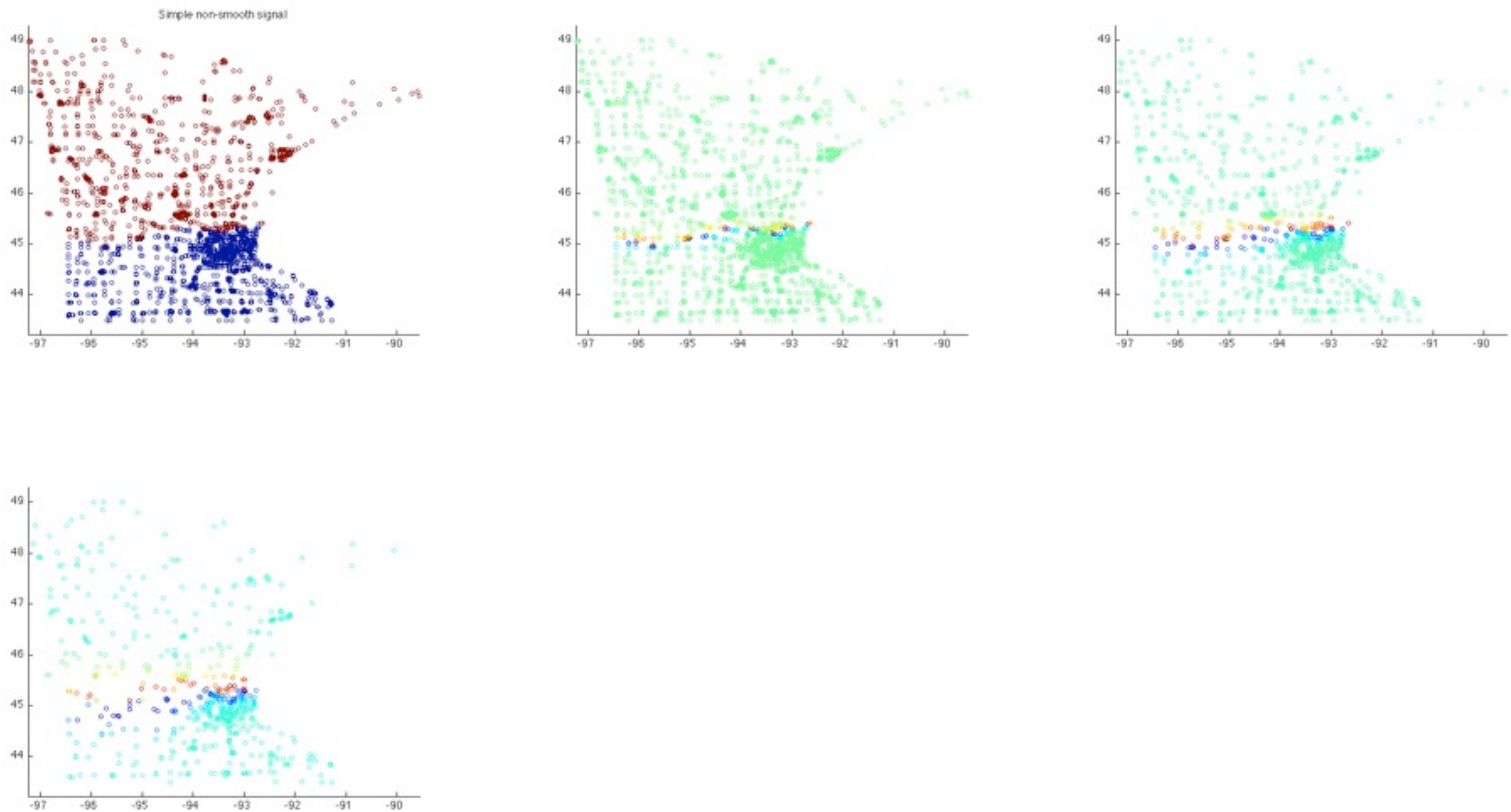
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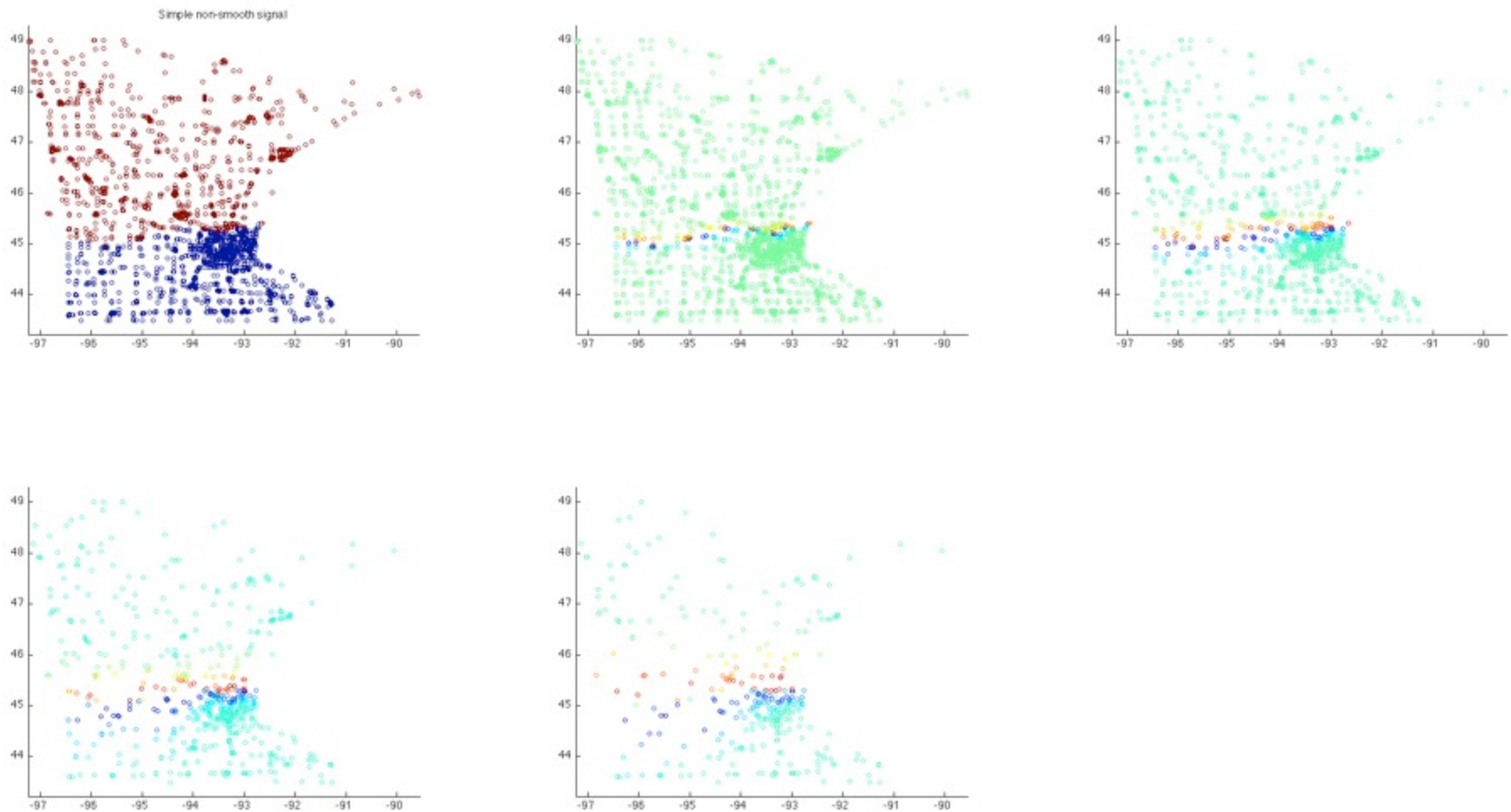
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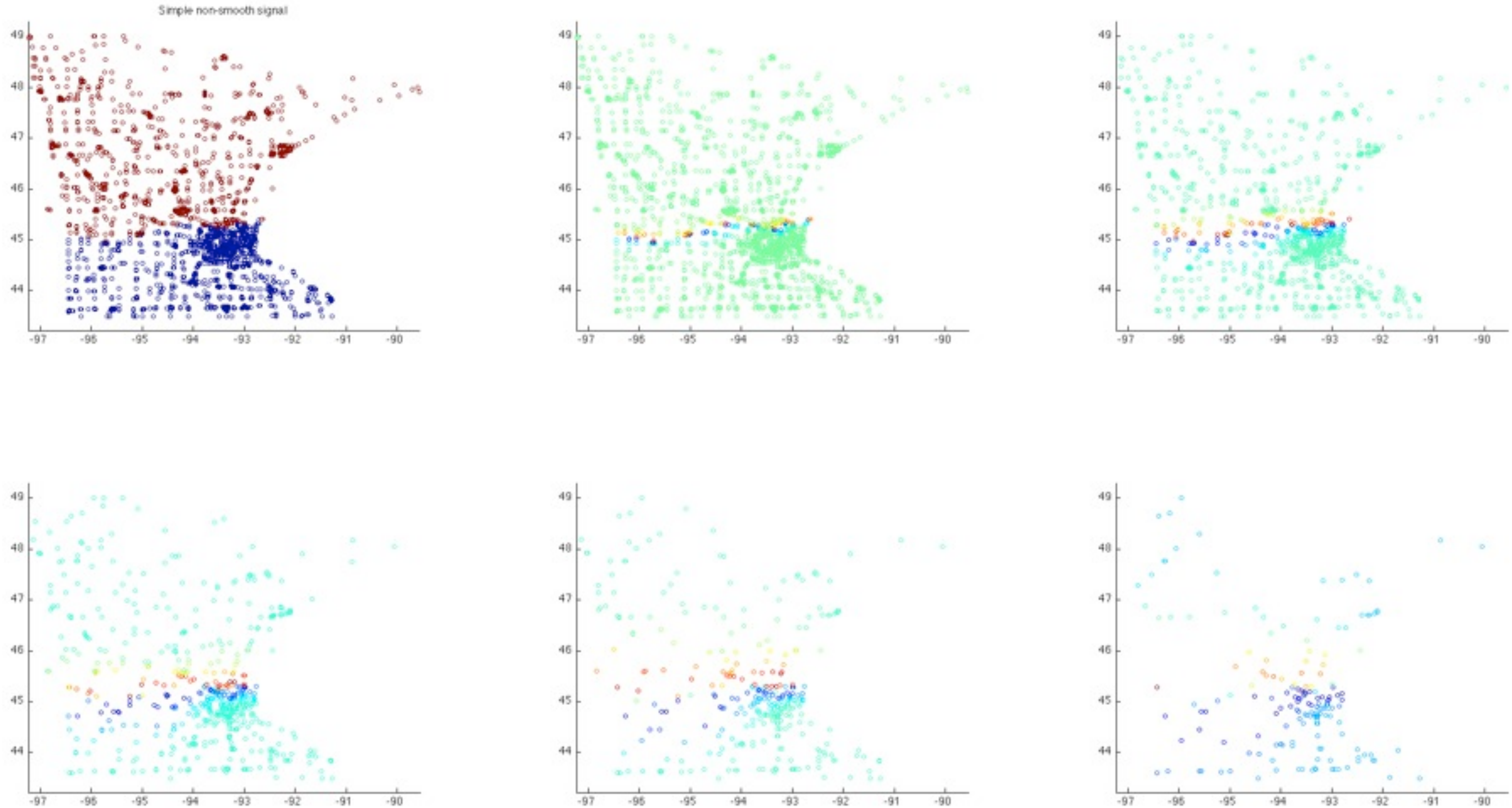
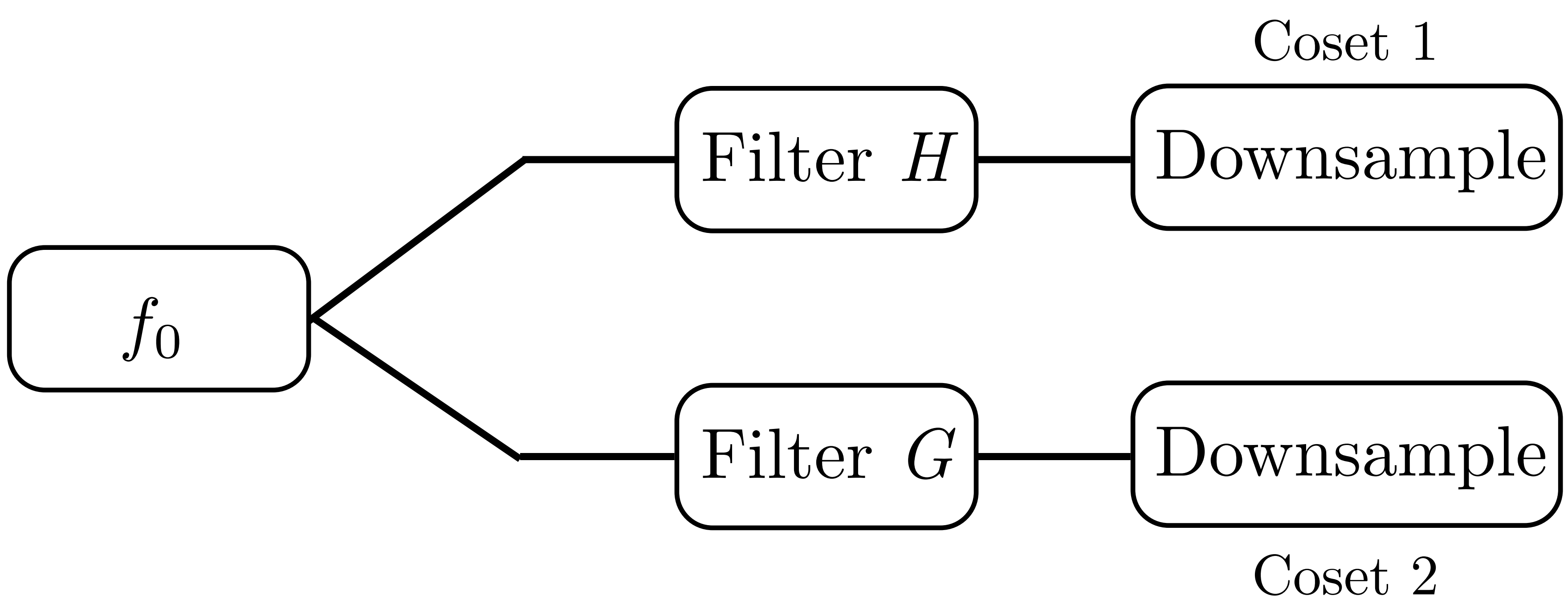

Simple non-smooth signal

Filter Banks

2 critically sampled channels



Filter Banks

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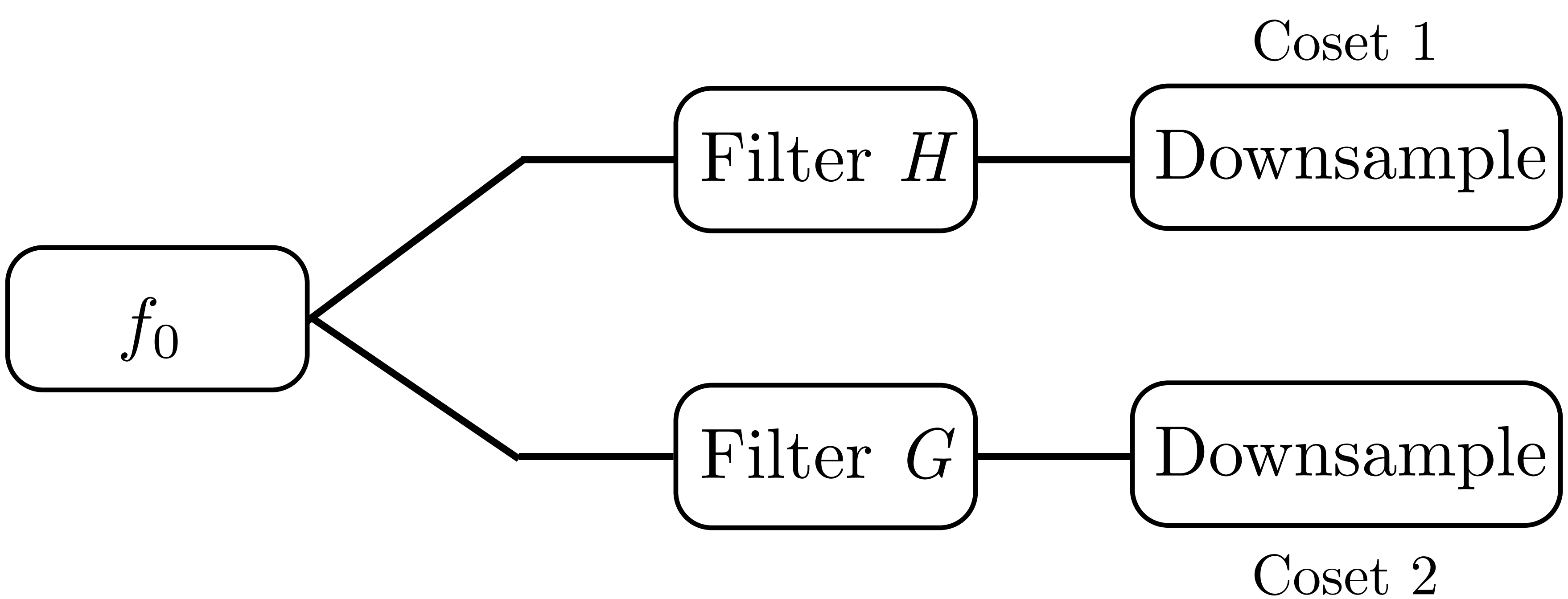

Theorem: For a k-RBG, the filter bank is perfect-reconstruction IFF

$$|H(i)|^2 + |G(i)|^2 = 2$$

$$H(i)G(N-i) + H(N-i)G(i) = 0$$

Conclusions

- Structured, data dependent dictionary of wavelets
  - sparsity and smoothness on graph are merged in simple and elegant fashion
  - fast algo, clean problem formulation
  - graph structure can be totally hidden in wavelets
- Filter banks based on nodal domains or coloring
  - Universal algo based on filtering and Kron reduction
  - Efficient IFF *some* structure in the graph
  - Unfortunately no closed form theory in general

Wavelet Ingredients

Wavelet transform based on two operations:

Dilation (or scaling) and Translation (or localization)

$$\psi_{s,a}(x) = \frac{1}{s}\psi\left(\frac{x-a}{s}\right)$$

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Equivalently: $$(T^s \delta_a)(x) = \frac{1}{s}\psi^*\left(\frac{x-a}{s}\right)$$

$$(T^s f)(x) = \frac{1}{2\pi}\int e^{i\omega x}\hat{\psi}^*(s\omega)\hat{f}(\omega)d\omega$$

Graph Laplacian and Spectral Theory

$G = (V, E, w)$ weighted, undirected graph

Non-normalized Laplacian: $\mathcal{L} = D - A$ Real, symmetric

$$(\mathcal{L}f)(i) = \sum_{i \sim j} w_{i,j}(f(i) - f(j))$$

Why Laplacian ?

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Why Laplacian ? $\mathbb{Z}^2$ with usual stencil

$$(\mathcal{L}f)_{i,j} = 4f_{i,j} - f_{i+1,j} - f_{i-1,j} - f_{i,j+1} - f_{i,j-1}$$

In general, graph laplacian from nicely sampled manifold converges to Laplace-Beltrami operator

Graph Laplacian and Spectral Theory

$$\frac{d^2}{dx^2} \quad \Rightarrow \quad e^{i\omega x} \quad \Rightarrow \quad f(x) = \frac{1}{2\pi}\int \hat{f}(\omega) e^{i\omega x} d\omega$$

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For simplicity assume connected graph and $0 = \lambda_0 < \lambda_1 \leq \lambda_2 ... \leq \lambda_{N-1}$

For any function on the vertex set (vector) we have:

$$\hat{f}(\ell) = \langle \phi_\ell, f \rangle = \sum_{i=1}^{N} \phi_\ell^*(i) f(i)$$ Graph Fourier Transform

$$f(i) = \sum_{\ell=0}^{N-1} \hat{f}(\ell)\phi_\ell(i)$$

Spectral Graph Wavelets

Remember good old Euclidean case:

$$(T^s f)(x) = \frac{1}{2\pi} \int e^{i\omega x} \hat{\psi}^*(s\omega) \hat{f}(\omega) d\omega$$

We will adopt this operator view

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We will adopt this operator view

Operator-valued function via continuous *Borel functional calculus*

$g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ $\qquad$ $T_g = g(\mathcal{L})$ **Operator-valued function**

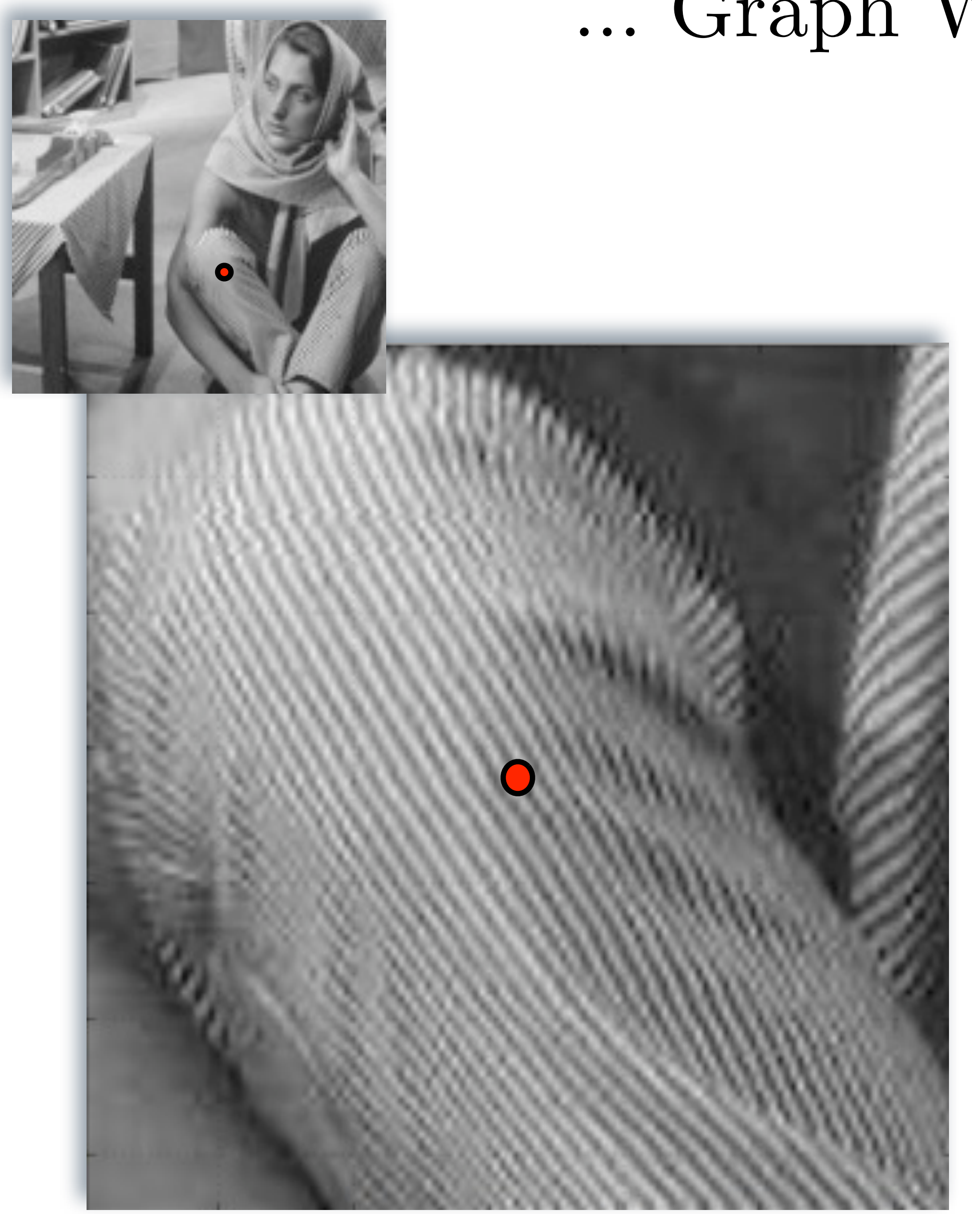
Action of operator is induced by its Fourier symbol

$$\widehat{T_g f}(\ell) = g(\lambda_\ell) \hat{f}(\ell) \qquad (T_g f)(i) = \sum_{\ell=0}^{N-1} g(\lambda_\ell) \hat{f}(\ell) \phi_\ell(i)$$

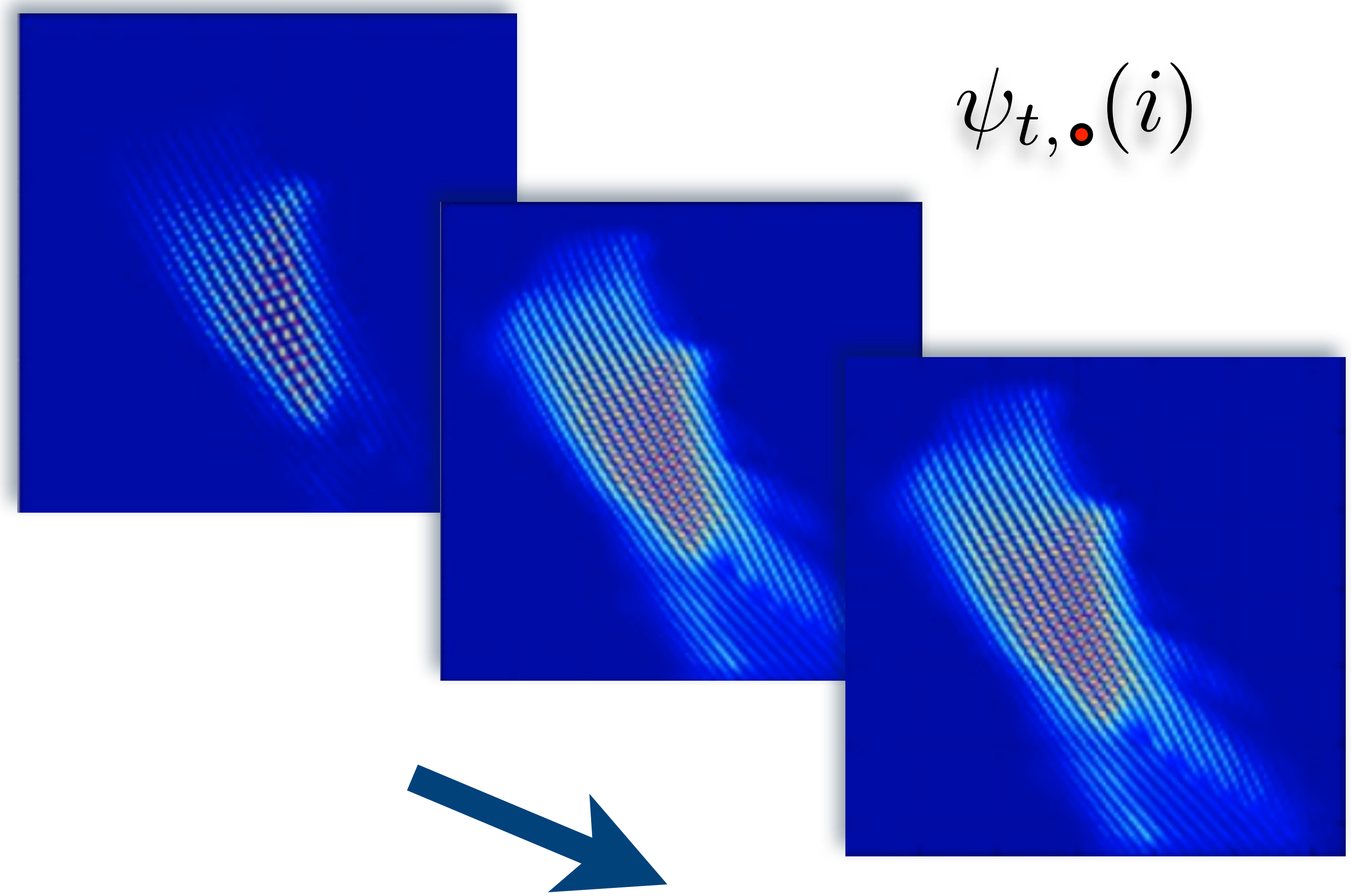
Non-local Wavelet Frame

- Non-local Wavelets are ...

... Graph Wavelets on Non-Local Graph



$\psi_{t,\bullet}(i)$



increasing scale

Interest: good *adaptive* sparsity basis

Distributed Computation

Scenario: Network of N nodes, each knows

- local data f(n)
- local neighbors
- M Chebyshev coefficients of wavelet kernel
- A global upper bound on largest eigenvalue of graph laplacian

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To compute: $$(\tilde{\mathbf{\Phi}} f)_{(j-1)N+n} = \Big(\frac{1}{2}c_{j,0}f + \sum_{k=1}^{M} c_{j,k}\overline{T}_k(\mathcal{L})f\Big)_n$$

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$$\left(\overline{T}_1(\mathcal{L}) f\right)_n = \left(\frac{2}{\alpha}(\mathcal{L} - \alpha I) f\right)_n$$

sensor only needs f(n) from its neighbors

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sensor only needs f(n) from its neighbors

$$\left(\overline{T}_k(\mathcal{L}) f\right) = \frac{2}{\alpha}(\mathcal{L} - \alpha I)\left(\overline{T}_{k-1}(\mathcal{L}) f\right) - \overline{T}_{k-2}(\mathcal{L}) f$$

Computed by exchanging last computed values

Distributed Computation

Communication cost: 2M|E| messages of length 1 per node

Example: distributed denoising, or distributed regression, with Lasso

$$\arg\min_a \frac{1}{2}\|y - \mathbf{\Phi}^* a\|_2^2 + \|a\|_{1,\mu}$$

$$a_i^{(k)} = \mathcal{S}_{\mu_i,\tau}\left(\left[a^{k-1} + \tau\mathbf{\Phi}(y - \mathbf{\Phi}^* a^{k-1})\right]_i\right)$$

$$\mathcal{S}_{\mu_i\tau}(z) := \begin{cases} 0 & , \text{ if } \mid z \mid\leq \mu_i\tau \\ z - \text{sgn}(z)\mu_i\tau & , \text{ o.w.} \end{cases}$$

Total communication cost:

Distributed Lasso [Mateos, Bazerque, Gianakis] Cost $\sim |E|N$

Chebyshev $\mathbf{\Phi} y$ 2M|E| messages of length 1

$\mathbf{\Phi\Phi}^* a$ 4M|E| messages of length J+1

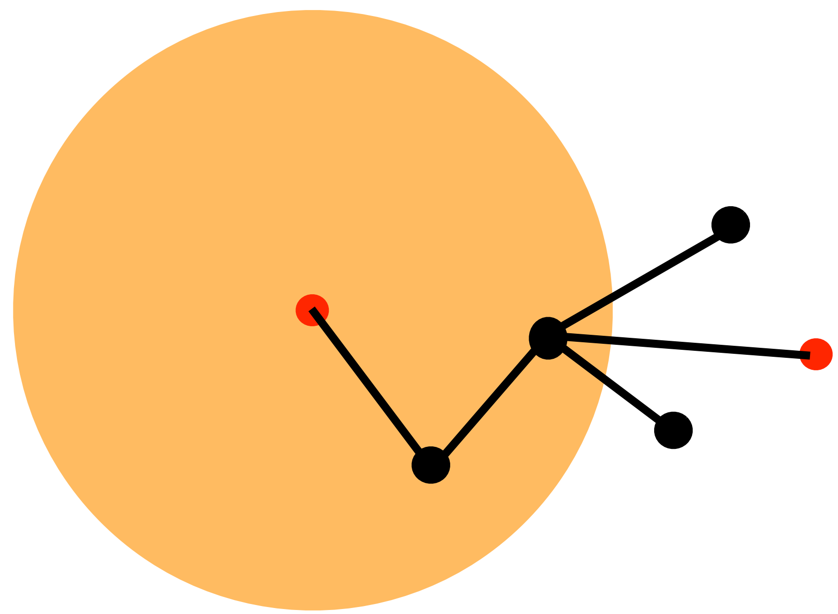
Cost $\sim |E|$

Wavelets on Graphs ?

- Existing constructions
  - wavelets on meshes (computer graphics, numerical analysis), often via lifting
  - diffusion wavelets [Maggioni, Coifman & others]
  - recently several other constructions based on "organizing" graph in a multiscale way [Gavish-Coifman]
- Goal
  - process signals on graphs
  - retain simplicity and signal processing flavor
  - algorithm to handle fairly large graphs

Scaling & Localization

Effect of operator dilation ?



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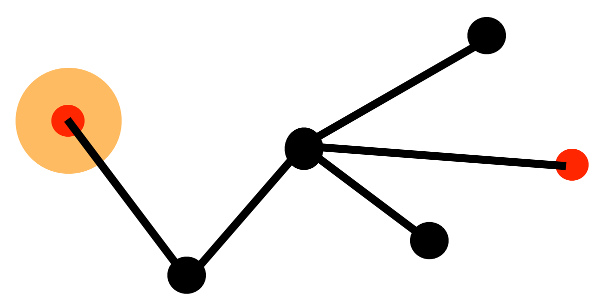

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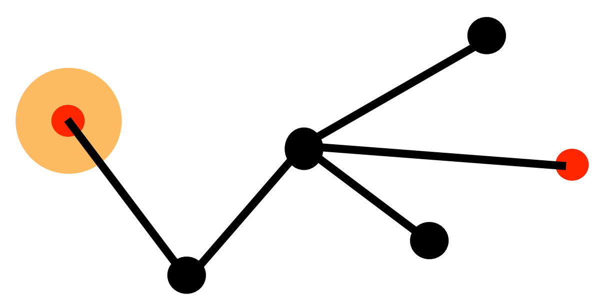

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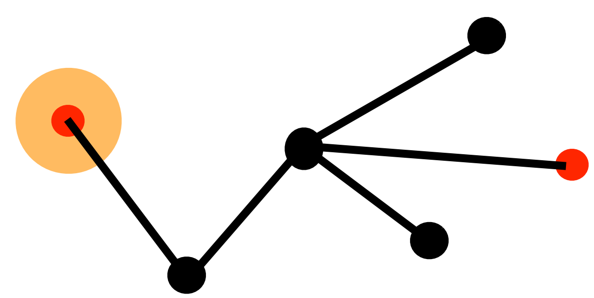

Theorem: $d_G(i,j) > K$ and g has K vanishing derivatives at *0*

$$\frac{\psi_{t,j}(i)}{\|\psi_{t,j}\|} \leq Dt$$ for any t smaller than a critical scale function of $d_G(i,j)$

Reason ? At small scale, wavelet operator behaves like power of Laplacian