DIFFUSE INTERFACE MODELS ON GRAPHS FOR CLASSIFICATION OF HIGH DIMENSIONAL DATA

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Thanks to J. Dobrosotskaya, S. Esedoglu, A. Flenner, A. Gillette, C. Schoenlieb

DIFFUSE INTERFACE METHODS - PDES

$$\int |\nabla u| dx \sim \frac{\epsilon}{2} \int |\nabla u|^2 + \frac{1}{\epsilon} \int W(u) dx$$

Total variation

Ginzburg-Landau functional

W is a double well potential with two minima

Total variation measures length of boundary between two constant regions.

GL energy is a diffuse interface approximation of TV for binary functionals

DIFFUSE INTERFACE EQUATIONS AND THEIR SHARP INTERFACE LIMIT

$$u_t = \epsilon \Delta u - \frac{1}{\epsilon} W'(u)$$

Allen-Cahn equation – L^2 gradient flow of GL functional
Approximates motion by mean curvaure - useful for image segmentation and image deblurring.

$$u_t = -\Delta(\epsilon \Delta u - \frac{1}{\epsilon} W'(u))$$

Cahn-Hilliard equation – H^{-1} gradient flow of GL functional
Approximates Mullins-Sekerka problem (nonlocal): Pego; Alikakos, Bates, and Chen. Conserves the mean of u.

Used in image inpainting – fourth order allows for two boundary conditions to be satisfied for inpainting.

EXAMPLES OF ALLEN-CAHN EQUATION IN IMAGE PROCESSING

$$E(u) = GL(u) + \lambda \int (K * u - f)^2 dx$$

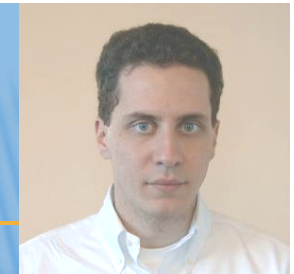
Selim Esedoglu : Blind deconvolution of bar codes *Inverse Problems* 2004 – K is a blurring kernel which can be identified as part of the process.
Uses a gradient flow method to minimize E. Results in a modified Allen-Cahn equation with forcing.

$$u_t = \epsilon \Delta u - \frac{1}{\epsilon} W'(u) \longrightarrow$$ Threshold Dynamics

Merriman-Bence-Osher show that AC can approximated by repeated thresholding and solution of heat equation- leads to a numerical solution of Motion by Mean Curvature.

Esedoglu-Tsai (*JCP 2006*) generalize this to the solution of the piecewise constant Mumford-Shah problem.

FAST CAHN-HILLIARD INPAINTING



US Patent No. 7,840,086

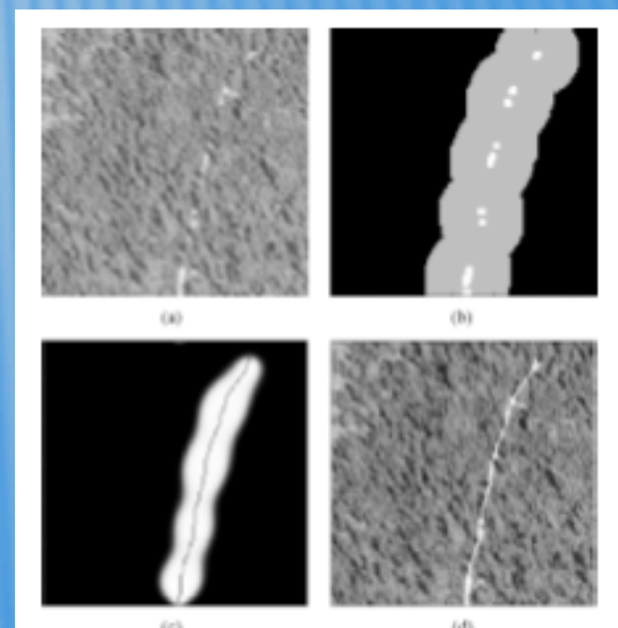
Bertozzi, Esedoglu, Gillette, *IEEE Trans. Image Proc. 2007, SIAM MMS 2007*

Transitioned to NGA for road inpainting.

Transitioned to InQtel for document exploitation.

Continue edges in the same direction – higher order method for local inpainting.

Fast method using convexity splitting and FFT



$$u_t = -\nabla^2 \left(\varepsilon \nabla^2 u - \frac{1}{\varepsilon} W'(u) \right) + \lambda(\vec{x})(f - u)$$

where

$$\lambda(\vec{x}) = \begin{cases} 0 & \text{if } \vec{x} \in D \\ \lambda_0 & \text{if } \vec{x} \in \Omega \setminus D. \end{cases}$$

H^{-1} gradient flow for diffuse TV

L^2 fidelity with known data

CONVEX SPLITTING SCHEMES – FAST METHODS

Bertozzi, Esedoglu, Gillette, *IEEE Trans. Imag. Proc*. 2007
Dobrosotskaya, Bertozzi, *IEEE Trans. Imag. Proc.* 2008
Schoenlieb and Bertozzi, *Comm. Math. Sci. 2011*

Basic idea:

$$E(u) = E_c(u) - E_e(u)$$

$$U_{k+1} - U_k = -\Delta t(\nabla E_c(U_{k+1}) - \nabla E_e(U_k))$$

Art is to choose E_c to give an implicit problem that is easy to solve
- e.g. E_c is H^1 semi norm – can be solved using FFT
- in wavelet case E_c is wavelet Laplace operator

Contraints on E_c and E_e so that splitting is unconditionally stable

Proof of convergence of splitting schemes for various higher order inpainting methods- with Carola Schoenlieb, CMS 2011

THE WAVELET LAPLACIAN AND DIFFUSE INTERFACES – SHARPER INTERFACES

$$\int |\nabla u| dx \sim \frac{\epsilon}{2} \int |\nabla u|^2 + \frac{1}{\epsilon} \int W(u) dx$$

Total variation Ginzburg-Landau functional

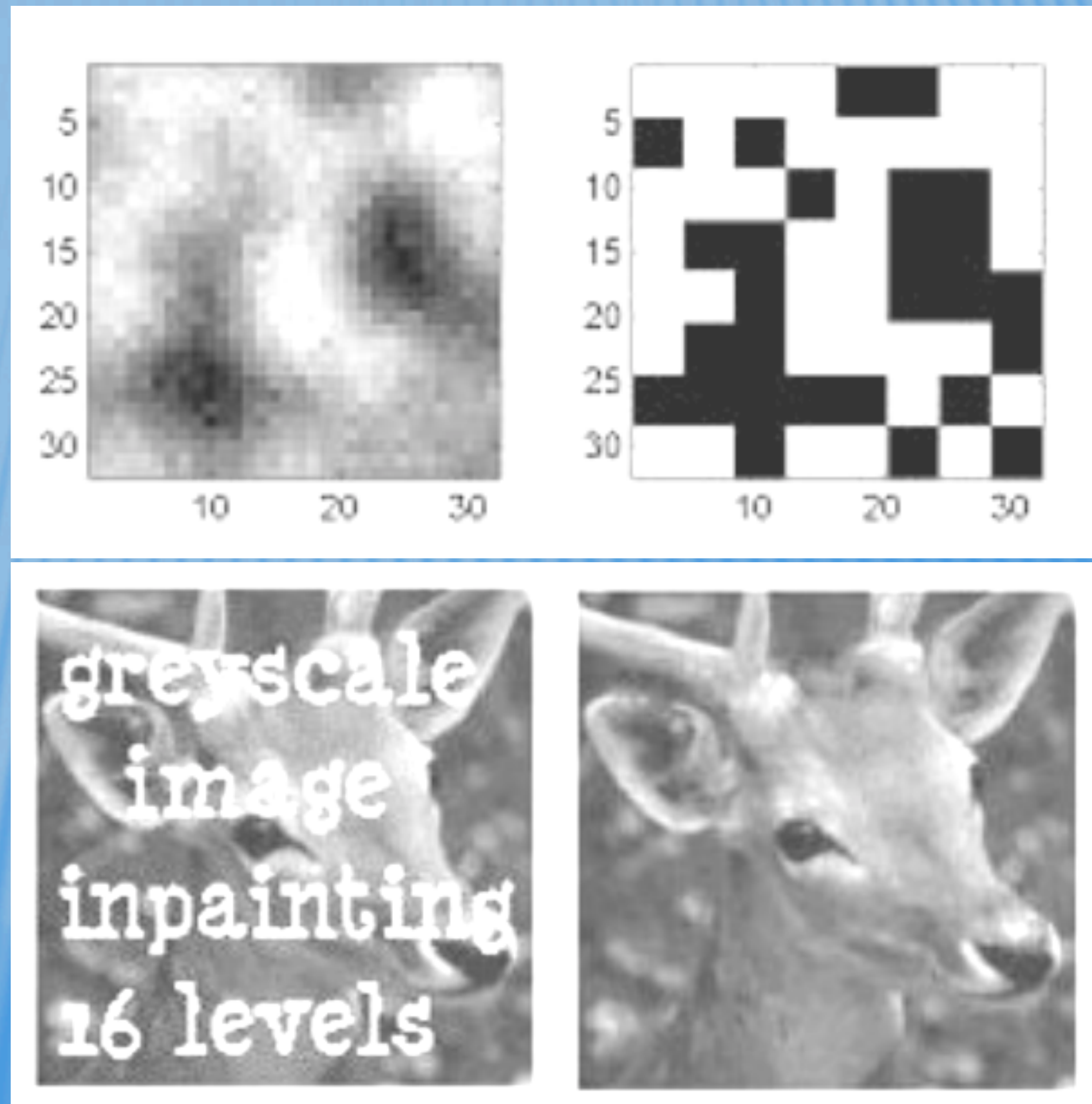
$$\Delta u \quad = \quad -4\pi^2 \sum_k k^2 < u, e^{-2\pi i k \cdot x} > e^{-2\pi i k \cdot x}$$

$$\Delta_w u \quad := \quad -\sum_j 2^{2j} < u, \psi_{j,k} > \psi_{j,k}$$

Dobrosotskaya and Bertozzi *IEEE Trans. Imag. Proc.* 2008

WAVELET ALLEN-CAHN IMAGE PROCESSING

- Dobrosotskaya, Bertozzi, *IEEE Trans. Image Proc. 2008, Interfaces and Free Boundaries 2011.*
- *Transitioned to NGA for road inpainting. Transitioned to InQtel for document exploitation.*
- Nonlocal wavelet basis replaces Fourier basis in classical diffuse interface method.
- Analysis theory in Besov spaces.
- Gamma convergence to anisotropic TV.



GAMMA CONVERGE OF WAVELET GINZBURG-LANDAU ENERGY

Dobrosotskaya and Bertozzi, *IFB* 2011

$$GL_\epsilon(f) = \frac{\epsilon}{2}\int |\nabla f(x)|^2 dx + \frac{1}{4\epsilon}\int W(f(x))dx, \;\; W(f) = (f^2-1)^2$$

$$WGL_\epsilon(f) = \frac{\epsilon}{2}|f|_B^2 + \frac{1}{4\epsilon}\int W(f(x))dx, f \in H^1,$$

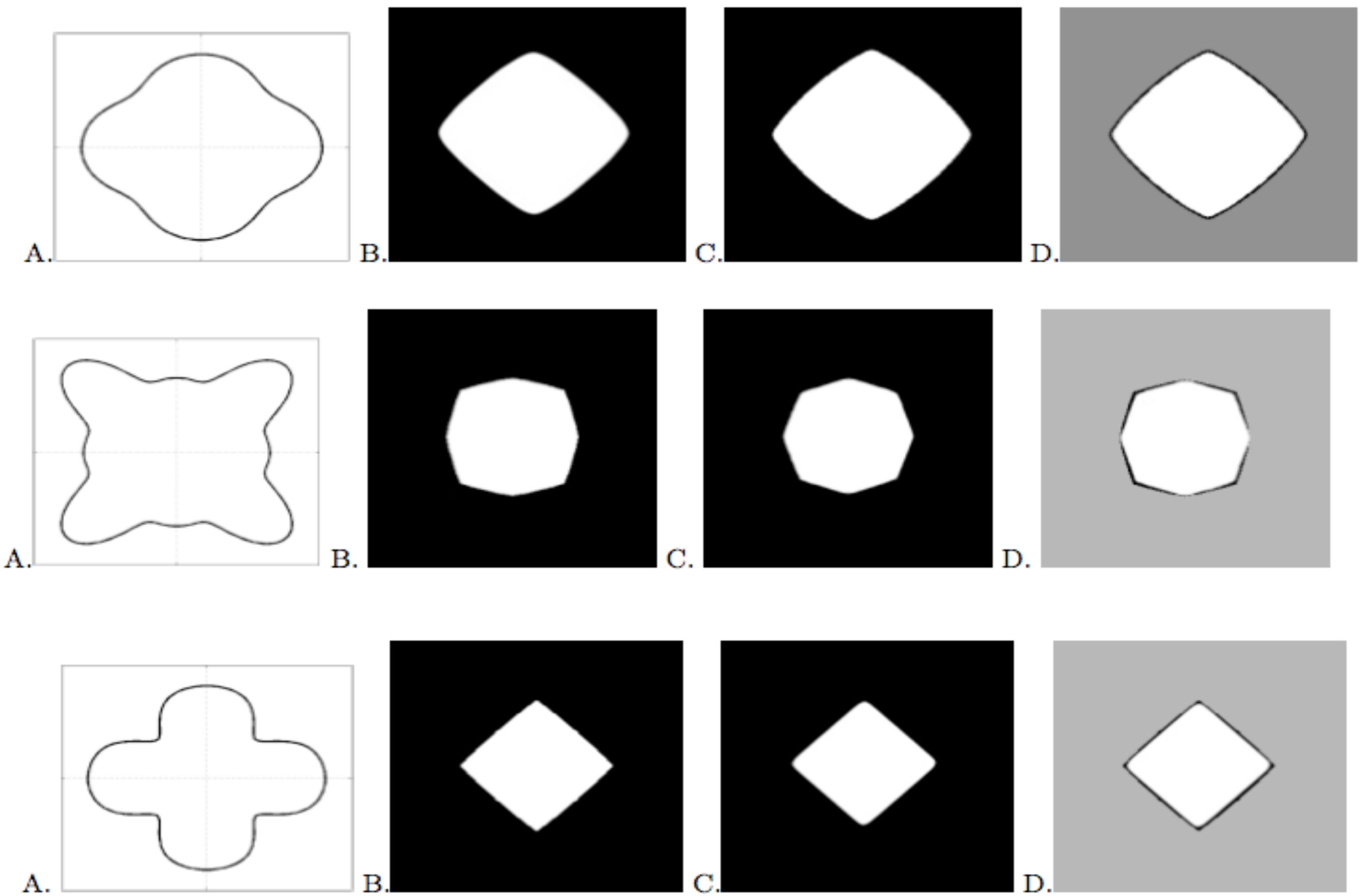
Theorem:

$$WGL_\epsilon(u_\epsilon) \xrightarrow{\Gamma} G_\infty(u), \;\; G_\infty(u) = \frac{\sqrt{2}}{3}C(u)|u|_{TV},$$

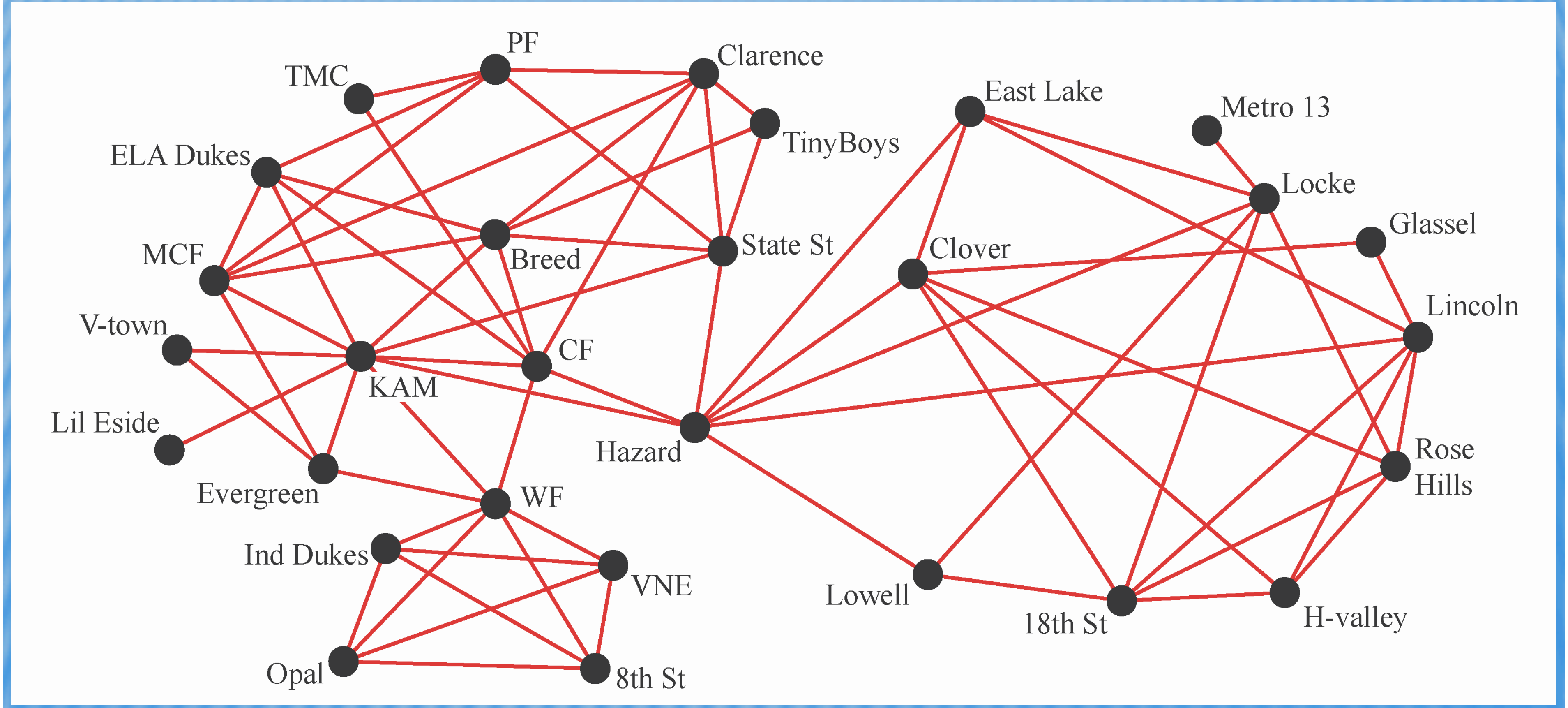
$$G_\infty(\chi_E) = \int_{\partial E} \rho(\vec{n}(x), \psi)dl(x),$$

GAMMA CONVERGE OF WAVELET GINZBURG-LANDAU ENERGY

Dobrosotskaya and Bertozzi, *IFB* 2011



DIFFUSE INTERFACES ON GRAPHS



Joint work with Arjuna Flenner, China Lake
Paper submitted to MMS 2011

HOW TO CHOOSE GRAPH WEIGHTS

$$w(x,y) = \exp(-||x-y||^2/\tau)$$

In a typical application we have data supported on the graph, possibly high dimensional. The above weights represent comparison of the data.

Examples include:

voting records of Congress – each person has a vote vector associated with them.

Nonlocal means image processing – each pixel has a pixel neighborhood that can be compared with nearby and far away pixels.

GRAPH BASED GL FUNCTIONAL

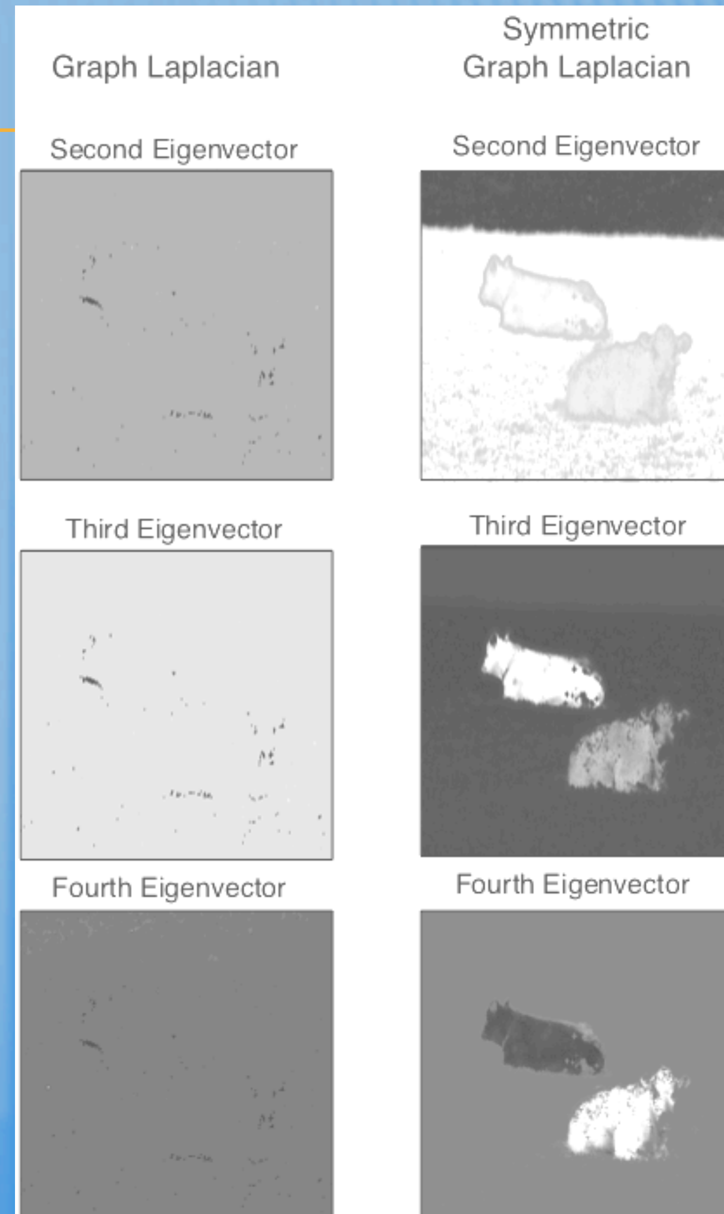
$$L(\nu,\mu) = \begin{cases} d(\nu) & \text{if } \nu = \mu, \\ -w(\nu,\mu) & \text{otherwise.} \end{cases}$$

$$\langle u, Lu \rangle = \frac{1}{2} \sum_{\mu,\nu \in V} w(\nu,\mu)(u(\nu) - u(\mu))^2$$

$$L_s = D^{-1/2} L D^{-1/2} = I - D^{-1/2} W D^{-1/2}.$$

$$\boxed{E(u) = \frac{\epsilon}{2}\langle u, L_s u \rangle + \frac{1}{4\epsilon}\sum_{z \in Z}(u^2(z) - 1)^2 + \sum_{z \in Z}\frac{\lambda(z)}{2}(u(z) - u_0(z))^2.}$$

PROPER NORMALIZATION OF GRAPH LAPLACIAN



This example uses nonlocal means-type weights between pixels

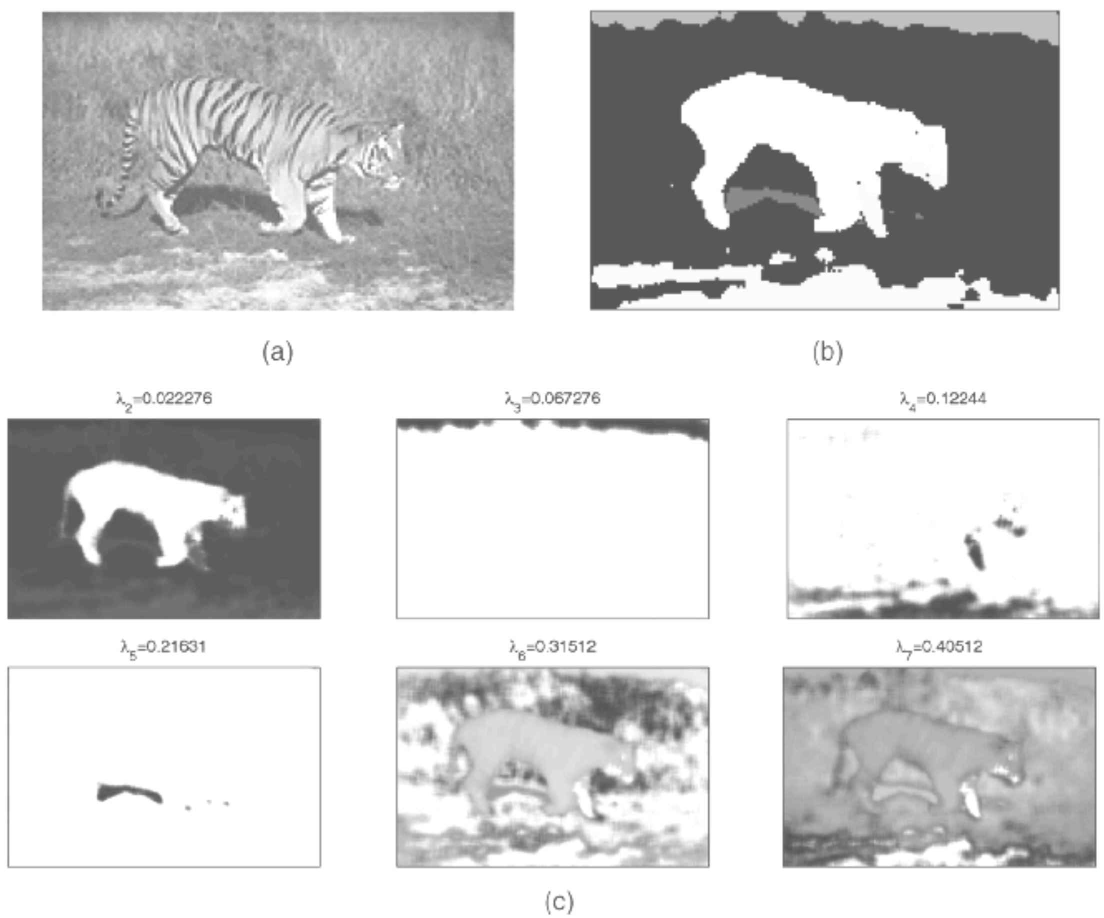
SPECTRAL GROUPING USING NYSTROM METHOD

Fowlkes, Belongie, Chung, and Malik, *IEEE TPAMI*, 2004

Application of a fast method for computing eigenfunctions of a linear operator – e.g. graph Laplacian.

Useful for fully connected graphs.

Example on right uses k-means clustering of eigenfunctions to segment the image.



CONVEX SPLITTING METHOD FOR GRAPHS

Convex Splitting for the Graph Laplacian

1. Input $\leftarrow$ an initial function u_0 and the eigenvalue-eigenvector pairs $(\tilde{\lambda}_k, \phi_k(x))$ for the graph Laplacian L_s from Equation (2.7).
2. Set convexity parameter c and interface scale ϵ from Equation (3.2).
3. Set the time step dt.
4. Initialize $a_k^{(0)} = \int u(x)\phi_k(x)\, dx$.
5. Initialize $b_k^{(0)} = \int [u_0(x)]^3 \phi_k(x)\, dx$.
6. Initialize $d_k^{(0)} = 0$.
7. Calculate $\mathcal{D}_k = 1 + dt\, (\epsilon\, \tilde{\lambda}_k + c)$.
8. For n less than a set number of iterations M
   (a) $a_k^{(n+1)} = \mathcal{D}_k^{-1} \left[\left(1 + \frac{dt}{\epsilon} + c\, dt\right)\, a_k^{(n)} - \frac{dt}{\epsilon} b_k^{(n)} - dt d_k^{(n)} \right]$
   (b) $u^{(n+1)}(x) = \sum_k a_k^{(n+1)} \phi_k(x)$
   (c) $b_k^{(n+1)} = \int [u^{(n+1)}(x)]^3 \phi_k(x)\, dx$
   (d) $d_k^{(n+1)} = \int \lambda(x) \left(u^{(n+1)}(x) - u_0(x)\right) \phi_k(x)\, dx$
9. end for
10. Output $\leftarrow$ the function $u^{(M)}(x)$.

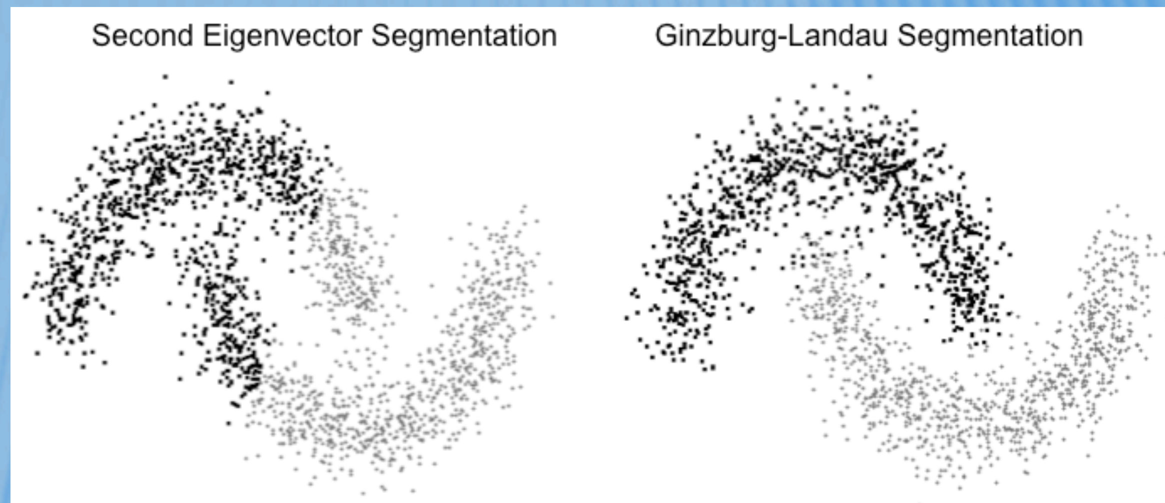
TWO MOONS EXAMPLE

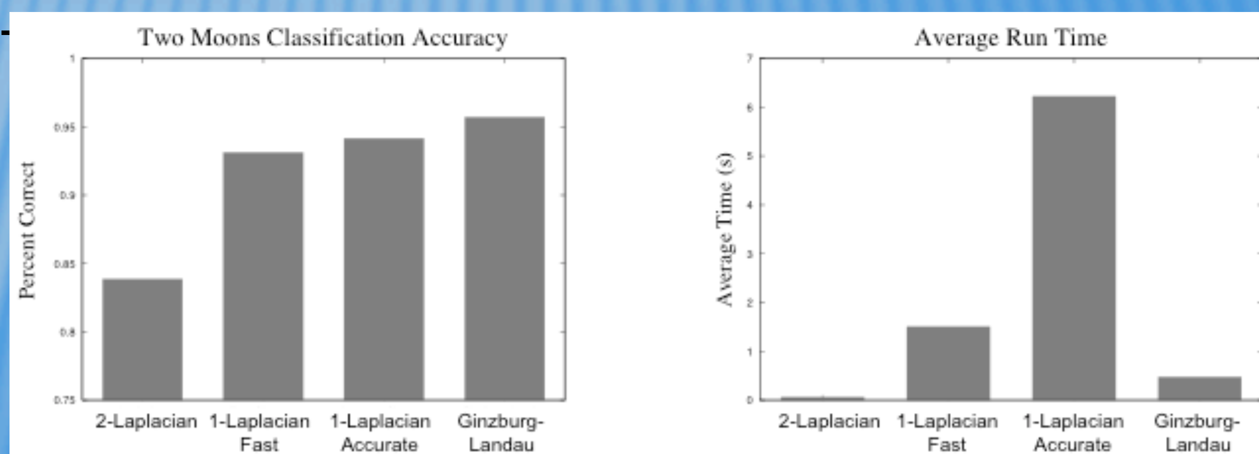
Data embedded in R^{100}

Replaces Laplace operator with a weighted graph Laplacian in the Ginzburg Landau Functional

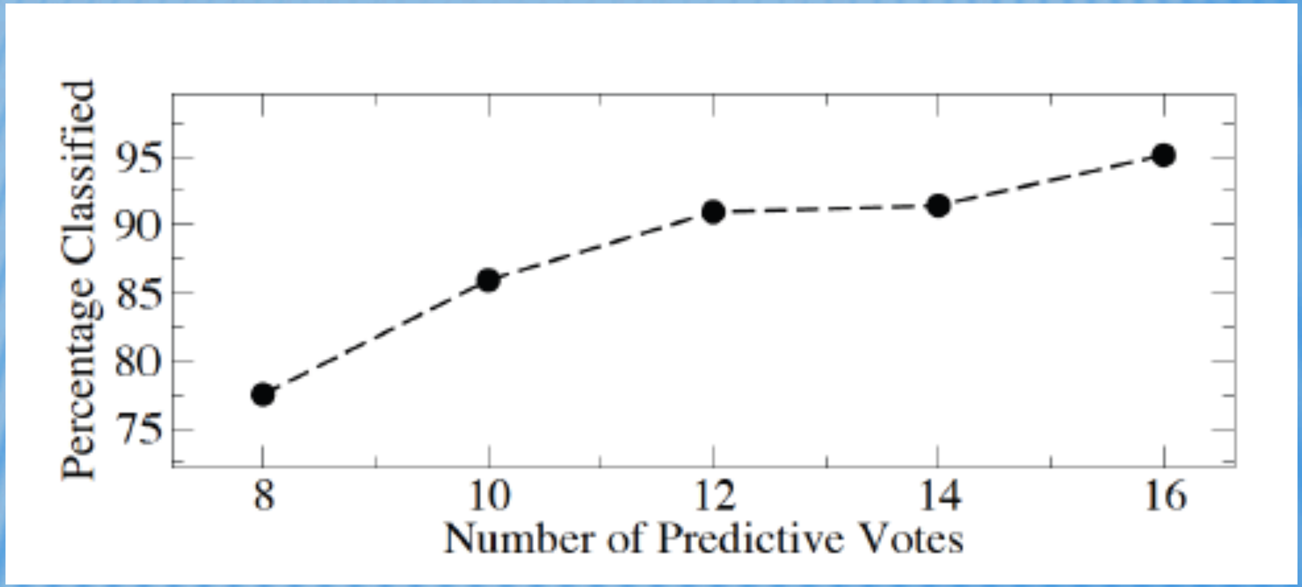
Allows for segmentation using L1-like metrics due to connection with G.

Compared with L1 methods of Hein and Beuhler NIPS 2010.





US HOUSE OF REPRESENTATIVES VOTING RECORD CLASSIFICATION OF PARTY AFFILIATION FROM VOTING RECORD



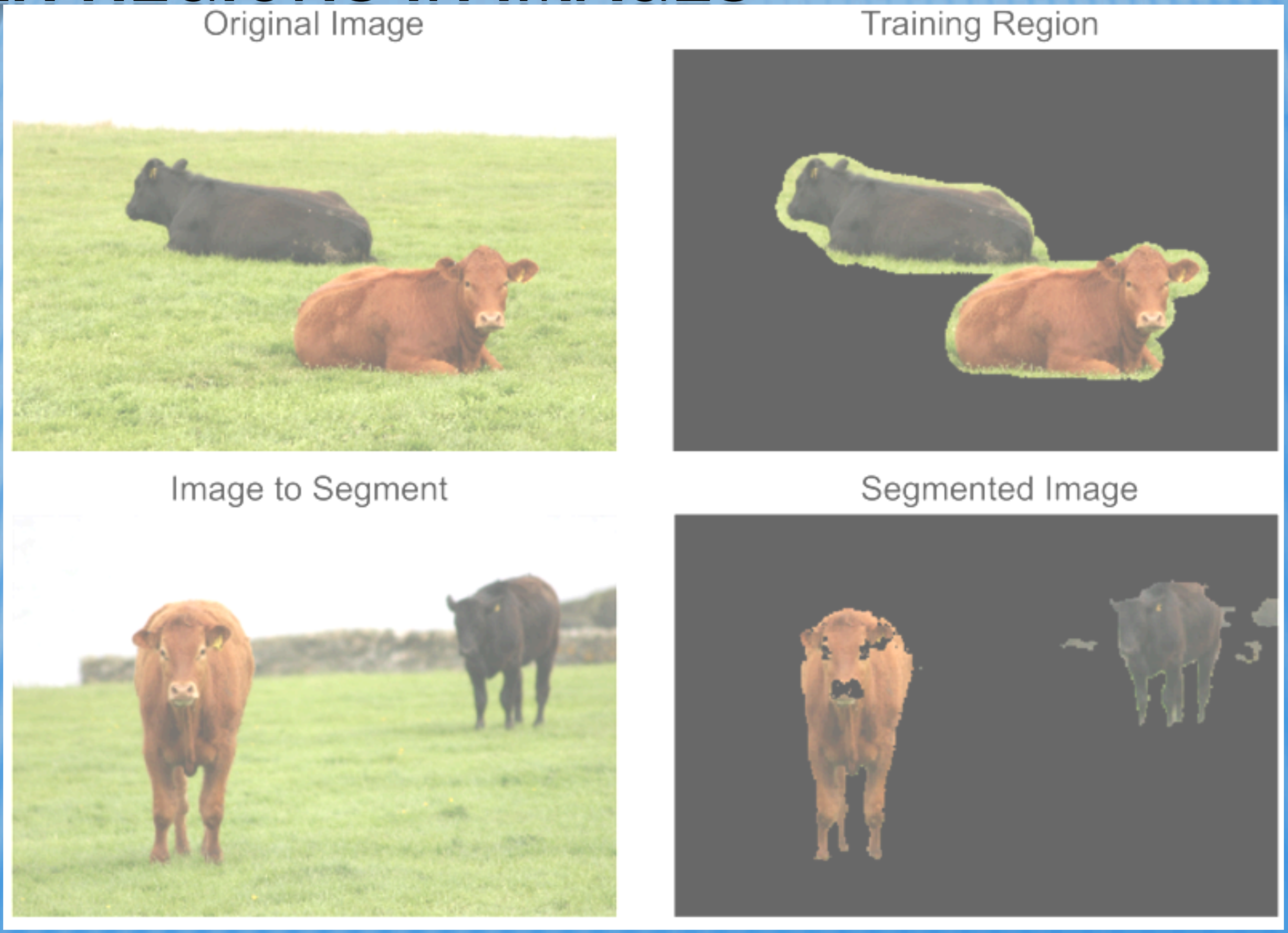
98th US Congress 1984

Assume knowledge of party affiliation of 5 of the 435 members of the House

Infer party affiliation of the remaining 430 members from voting records

Gaussian similarity weight matrix for vector of votes (1, 0, -1)

MACHINE LEARNING IDENTIFICATION OF SIMILAR REGIONS IN IMAGES



High dimensional fully connected graph – use Nystrom extension methods for fast computation methods.

PAPERS AND PREPRINTS

Andrea L. Bertozzi and Arjuna Flenner,
Diffuse interface models on graphs for classification of high dimensional data
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Julia A. Dobrosotskaya and Andrea L. Bertozzi,
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Interfaces and Free Boundaries, 12(2), 2010, pp. 497-525.

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A Wavelet-Laplace Variational Technique for Image Deconvolution and Inpainting,
IEEE Trans. Imag. Proc., 17(5), pages 657-663, 2008.

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Analysis of a two-scale Cahn-Hilliard model for image inpainting,
Multiscale Modeling and Simulation, vol. 6, no. 3, pages 913-936, 2007.

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IEEE Trans. Image Proc. 16(1) pp. 285-291, 2007.